Convergence of a finite-volume scheme for a heat equation with a multiplicative Lipschitz noise

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The heat equation with multiplicative Lipschitz noise

For $T > 0$, $\Lambda \subset \mathbb{R}^2$ a bounded polygonal domain we consider

$$du - \Delta u \, dt = g(u) \, dW_t \quad \text{in } \Omega \times (0, T) \times \Lambda$$

- $(\Omega, \mathcal{A}, \mathbb{P}, (\mathcal{F}_t)_{t \geq 0}, (W_t)_{t \geq 0})$ is a stochastic basis with a real-valued Brownian motion $(W_t)_{t \geq 0}$.
- $g : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous
- homogeneous Neumann boundary condition

$$\nabla u \cdot \vec{n} = 0$$

- initial value $u(0, \cdot) = u_0$ for a $\mathcal{F}_0$-measurable random variable $u_0 \in L^2(\Omega; H^1(\Lambda))$.

Some useful facts on stochastic analysis

- A filtration $(\mathcal{F}_t)_{t\geq 0}$ is a family of σ-fields $(\mathcal{F}_t)_{t\geq 0}$ satisfying $\mathcal{F}_t \subseteq \mathcal{A}$ for all $t \geq 0$ and $\mathcal{F}_s \subseteq \mathcal{F}_t$ for all $s \leq t$.

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- A stochastic process $(W_t)_{t\geq 0}$ is a Brownian motion with respect to $(\mathcal{F}_t)_{t\geq 0}$ iff
  - $W_0 = 0$
  - For any fixed $t \in [0, T]$, the random variable W_t is $\mathcal{F}_t$-measurable, i.e., $(\mathcal{F}_t)_{t\geq 0}$-*adapted*
  - For for $0 \leq s \leq t$, $W_t - W_s$ is $N(0, t-s)$-distributed and independent of $\mathcal{F}_s$

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- Properties of the Itô integral

$$\mathbb{E}\left[\int_0^T \phi(s)\, dW_s\right] = 0$$

$$\mathbb{E}\left[\left\|\int_0^T \phi(s)\, dW_s\right\|^2\right] = \mathbb{E}\left[\int_0^T \|\phi(s)\|^2\, ds\right]$$

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Finite-volume approximation of the variational solution

A variational solution to the heat equation with multiplicative Lipschitz noise is a $(\mathcal{F}_t)_{t\geq 0}$-adapted stochastic process

$$u \in L^2(\Omega; \mathcal{C}([0,T]; L^2(\Lambda))) \cap L^2(\Omega; L^2(0,T; H^1(\Lambda)))$$

such that, for all $t \in [0,T]$, in $L^2(\Lambda)$, $\mathbb{P}$-a.s. in Ω,

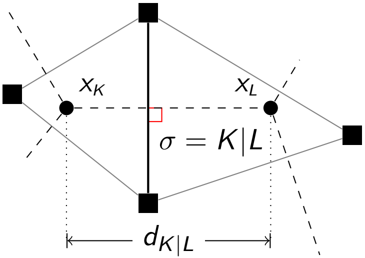
$$u(t) - u_0 - \int_0^t \Delta u(s)\, ds = \int_0^t g(u(s))\, dW_s$$

- From classical results (see, e.g., [Pardoux 1975], [Krylov, Rozovskii 1981], [Liu, Röckner 2015],...) existence and uniqueness of a *variational solution* is well-known.

We propose a finite-volume scheme, semi-implicit in time and a **T**wo-**P**oint **F**lux **A**pproximation (TPFA) in space and show its convergence to the variational solution.

The mesh on Λ

Let $\mathcal{T}$ be an admissible mesh consisting of open, polygonal and convex subsets, i.e., control volumes $K \in \mathcal{T}$



- To each $K \in \mathcal{T}$ we associate a point $x_K \in K$, called center
- $\sigma = K|L$ is the interface between two neighbouring control volumes $K, L \in \mathcal{T}$, called edge
- For two neighbouring control volumes $K, L \in \mathcal{T}$ with centers x_K, x_L we have the orthogonality condition

$$\overrightarrow{x_k x_L} \perp K|L$$

TPFA for the Laplace operator

For $u \in \mathcal{C}^2(\overline{\Lambda})$ with $\nabla u \cdot \vec{\mathrm{n}} = 0$, $K \in \mathcal{T}$, by the divergence theorem

$$\int_K \Delta u \, dx = \oint_{\partial K} \nabla u \cdot \vec{\mathrm{n}}_K \, dS$$

TPFA for the Laplace operator

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$$\int_K \Delta u \, dx = \oint_{\partial K} \nabla u \cdot \vec{\mathsf{n}}_K \, dS = \sum_{\sigma \in \mathcal{E}_{int} \cap \mathcal{E}_K} \oint_\sigma \nabla u \cdot \vec{\mathsf{n}}_{K|L} \, dS$$

- $\mathcal{E}_K :=$ set of edges of K for $K \in \mathcal{T}$
- $\mathcal{E}_{int} :=$ set of all *interior edges* of $\mathcal{T}$
- on $\sigma \in \mathcal{E}_{int}$, $\vec{\mathsf{n}}_K = \vec{\mathsf{n}}_{K|L}$ pointing towards a neighbouring control volume $L \in \mathcal{T}$

TPFA for the Laplace operator

For $u \in \mathcal{C}^2(\overline{\Lambda})$ with $\nabla u \cdot \vec{\mathrm{n}} = 0$, $K \in \mathcal{T}$ by the divergence theorem and the orthogonality condition on $\mathcal{T}$

$$\begin{aligned}\int_K \Delta u \, dx &= \oint_{\partial K} \nabla u \cdot \vec{\mathrm{n}}_K \, dS \\ &= \sum_{\sigma \in \mathcal{E}_{int} \cap \mathcal{E}_K} \oint_\sigma \nabla u \cdot \vec{\mathrm{n}}_{K|L} \, dS \\ &= \sum_{\sigma \in \mathcal{E}_{int} \cap \mathcal{E}_K} \oint_\sigma \nabla u \cdot \frac{(x_L - x_K)}{d_{K|L}} \, dS\end{aligned}$$

- $\mathcal{E}_K :=$ set of edges of K for $K \in \mathcal{T}$
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- $\vec{\mathrm{n}}_K = \vec{\mathrm{n}}_{K|L}$ pointing towards $L \in \mathcal{T}$, $d_{K|L} := |x_K - x_L|$.

TPFA for the Laplace operator

For $u \in \mathcal{C}^2(\overline{\Lambda})$ with $\nabla u \cdot \vec{n} = 0$, $K \in \mathcal{T}$ by the divergence theorem, the orthogonality condition on $\mathcal{T}$ and Taylor expansion

$$\begin{aligned}
\int_K \Delta u \, dx &= \oint_{\partial K} \nabla u \cdot \vec{n}_K \, dS \\
&= \sum_{\sigma \in \mathcal{E}_{int} \cap \mathcal{E}_K} \oint_\sigma \nabla u \cdot \vec{n}_{K|L} \, dS \\
&= \sum_{\sigma \in \mathcal{E}_{int} \cap \mathcal{E}_K} \oint_\sigma \nabla u \cdot \frac{(x_L - x_K)}{d_{K|L}} \, dS \\
&= \sum_{\sigma \in \mathcal{E}_{int} \cap \mathcal{E}_K} (u_L - u_K) \frac{m_\sigma}{d_{K|L}} + o(\text{size}(\mathcal{T}))
\end{aligned}$$

- $\mathcal{E}_K :=$ set of edges of K for $K \in \mathcal{T}$
- $\mathcal{E}_{int} :=$ set of all *interior edges* of $\mathcal{T}$
- $\vec{n}_K = \vec{n}_{K|L}$ pointing towards $L \in \mathcal{T}$, $d_{K|L} := |x_K - x_L|$.
- $u_K := u(x_K)$, $u_L := u(x_L)$, $m_\sigma :=$ length of σ

TPFA for the Laplace operator

For $u \in \mathcal{C}^2(\overline{\Lambda})$, with $\nabla u \cdot \vec{\mathrm{n}} = 0$ and $K \in \mathcal{T}$ by the divergence theorem, the orthogonality condition on $\mathcal{T}$ and Taylor expansion

$$
\begin{aligned}
\int_K \Delta u \, dx &= \oint_{\partial K} \nabla u \cdot \vec{\mathrm{n}}_K \, dS \\
&= \sum_{\sigma \in \mathcal{E}_{int} \cap \mathcal{E}_K} \oint_\sigma \nabla u \cdot \vec{\mathrm{n}}_{K|L} \, dS \\
&= \sum_{\sigma \in \mathcal{E}_{int} \cap \mathcal{E}_K} \oint_\sigma \nabla u \cdot \frac{(x_L - x_K)}{d_{K|L}} \, dS \\
&= \underbrace{\sum_{\sigma \in \mathcal{E}_{int} \cap \mathcal{E}_K} (u_L - u_K) \frac{m_\sigma}{d_{K|L}}}_{\text{two point flux approximation for } \Delta u} + o(\mathrm{size}(\mathcal{T}))
\end{aligned}
$$

The finite-volume scheme

Proposition [Bauzet, Nabet, Schmitz, Z., '22]

For $h > 0$, let $\mathcal{T}_h$ be an admissible mesh with $\text{size}(\mathcal{T}_h) = h$ and $m_K := |K|$ for $K \in \mathcal{T}_h$. For $N \in \mathbb{N}$, let $\triangle t := \frac{T}{N}$ and $t_n := n\triangle t$ for $n = 0, \dots, N$.

For any given $\mathcal{F}_{t_n}$-measurable random vector $(u_K^n)_{K\in\mathcal{T}_h}$, there exists a $\mathcal{F}_{t_{n+1}}$-measurable random vector $(u_K^{n+1})_{K\in\mathcal{T}_h}$ satisfying

$$m_K(u_K^{n+1} - u_K^n) + \triangle t \sum_{\sigma\in\mathcal{E}_{int}\cap\mathcal{E}_K} \frac{m_\sigma}{d_{K|L}}(u_K^{n+1} - u_L^{n+1}) = m_K g(u_K^n)(W_{t_{n+1}} - W_{t_n}) \qquad \text{(FV)}$$

for all $K \in \mathcal{T}_h$, $\mathbb{P}$-a.s. in Ω.

Main result

For $h > 0$, let $\mathcal{T}_h$ be an admissible mesh with $\text{size}(\mathcal{T}_h) = h$. For $N \in \mathbb{N}$, let $\triangle t := \frac{T}{N}$ and $t_n := n\triangle t$ for $n = 0, \dots, N$. For any $K \in \mathcal{T}_h$ let

$$u_K^0 := \frac{1}{m_K} \int_K u_0(x)\, dx.$$

For $n \in \{0, \dots, N-1\}$, let $(u_K^{n+1})_{K \in \mathcal{T}_h}$ be the solution of (FV) obtained by iteration starting with the random vector $(u_K^0)_{K \in \mathcal{T}_h}$.

Then, the step functions

$$u_{h,N}^r(t,x) := u_K^{n+1},\ \ t \in [t_n, t_{n+1}),\ \ x \in K \ \textbf{(not adapted)}$$

$$u_{h,N}^l(t,x) := u_K^{n},\ \ t \in [t_n, t_{n+1}),\ \ x \in K \ \text{(adapted)}$$

converge in $L^p(\Omega, L^2(0,T; L^2(\Lambda)))$ for all $1 \leq p < 2$ towards the unique variational solution of the heat equation with multiplicative Lipschitz noise.

Fundamental inequality

$$m_K(u_K^{n+1} - u_K^n) + \triangle t \sum_{\sigma \in \mathcal{E}_{int} \cap \mathcal{E}_K} \frac{m_\sigma}{d_{K|L}} (u_K^{n+1} - u_L^{n+1})$$
$$= m_K g(u_K^n)(W_{t_{n+1}} - W_{t_n})$$

Fundamental inequality

$$\sum_{K\in\mathcal{T}_h} m_K(u_K^{n+1}-u_K^n)u_K^{n+1}+\triangle t\sum_{K\in\mathcal{T}_h}\sum_{\sigma\in\mathcal{E}_{int}\cap\mathcal{E}_K}\frac{m_\sigma}{d_{K|L}}(u_K^{n+1}-u_L^{n+1})u_K^{n+1}$$
$$=\sum_{K\in\mathcal{T}_h} m_K g(u_K^n)(W_{t_{n+1}}-W_{t_n})u_K^{n+1}$$

Fundamental inequality

$$\sum_{K\in\mathcal{T}_h} m_K(u_K^{n+1}-u_K^n)u_K^{n+1} + \triangle t \overbrace{\sum_{K\in\mathcal{T}_h}\sum_{\sigma\in\mathcal{E}_{int}\cap\mathcal{E}_K}\frac{m_\sigma}{d_{K|L}}(u_K^{n+1}-u_L^{n+1})u_K^{n+1}}^{=\sum_{\sigma\in\mathcal{E}_{\text{int}}}\frac{m_\sigma}{d_{K|L}}|u_K^{n+1}-u_L^{n+1}|^2}$$

$$= \sum_{K\in\mathcal{T}_h} m_K g(u_K^n)(W_{t_{n+1}}-W_{t_n})u_K^{n+1}$$

Fundamental inequality

$$\sum_{K\in\mathcal{T}_h} \frac{m_K}{2}\mathbb{E}\left[|u_K^{n+1}|^2 - |u_K^n|^2 + |u_K^{n+1} - u_K^n|^2\right]$$

$$+ \triangle t\mathbb{E}\left[\sum_{\sigma\in\mathcal{E}_{\text{int}}} \frac{m_\sigma}{d_{K|L}}|u_K^{n+1} - u_L^{n+1}|^2\right]$$

$$= \sum_{K\in\mathcal{T}_h} m_K\mathbb{E}\left[g(u_K^n)(W_{t_{n+1}} - W_{t_n})u_K^{n+1}\right]$$

Fundamental inequality

$$\sum_{K\in\mathcal{T}_h}\frac{m_K}{2}\mathbb{E}\left[|u_K^{n+1}|^2-|u_K^n|^2+|u_K^{n+1}-u_K^n|^2\right]$$
$$+\triangle t\mathbb{E}\left[\sum_{\sigma\in\mathcal{E}_{\text{int}}}\frac{m_\sigma}{d_{K|L}}|u_K^{n+1}-u_L^{n+1}|^2\right]$$
$$=\sum_{K\in\mathcal{T}_h}m_K\mathbb{E}\left[g(u_K^n)(W_{t_{n+1}}-W_{t_n})(u_K^{n+1}-u_K^n)\right]$$

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$$+\triangle t\mathbb{E}\left[\sum_{\sigma\in\mathcal{E}_{\text{int}}}\frac{m_\sigma}{d_{K|L}}|u_K^{n+1}-u_L^{n+1}|^2\right]$$

$$\leq\sum_{K\in\mathcal{T}_h}\frac{m_K}{2}\left(\triangle t\mathbb{E}\left[|g(u_K^n)|^2\right]+\mathbb{E}\left[|u_K^{n+1}-u_K^n|^2\right]\right)$$

Fundamental inequality

$$\sum_{K\in\mathcal{T}_h}\frac{m_K}{2}\mathbb{E}\left[|u_K^{n+1}|^2-|u_K^n|^2\right]+\mathbb{E}\left[\int_{t_n}^{t_{n+1}}\sum_{\sigma\in\mathcal{E}_{\text{int}}}\frac{m_\sigma}{d_{K|L}}|u_K^{n+1}-u_L^{n+1}|^2\,dt\right]$$
$$\leq\sum_{K\in\mathcal{T}_h}\frac{m_K}{2}\mathbb{E}\left[\int_{t_n}^{t_{n+1}}|g(u_K^n)|^2\,dt\right]$$

for all $n\in\{0,\dots,N-1\}$.

Consequences of the fundamental inequality

Using the Lipschitz continuity of $g : \mathbb{R} \to \mathbb{R}$ and a discrete Gronwall inequality, it follows that

- the sequences of step functions $(u^l_{h,N})_{h,N}$ and $(u^r_{h,N})_{h,N}$
- the norm of the *discrete gradient*, i.e.,

$$\|\nabla^h u^r_{h,N}\|^2 = 2\mathbb{E}\left[\sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}} \sum_{\sigma\in\mathcal{E}_{\text{int}}} \frac{m_\sigma}{d_{K|L}}|u_K^{n+1} - u_L^{n+1}|^2 \, dt\right]$$

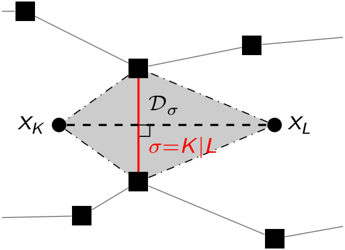
are bounded in $L^2(\Omega; L^2(0, T; L^2(\Lambda)))$ by constants not depending on the discretization parameters $N \in \mathbb{N}$ and $h > 0$.

The discrete gradient

For $t \in [t_n, t_{n+1})$, $n \in \{0, \dots, N-1\}$ we associate to the step function $u^r_{h,N}$ given by (FV) a discrete gradient

$$\nabla^h u^r_{h,N}(t,x) = \begin{cases} 2\dfrac{u_L^{n+1} - u_K^{n+1}}{d_{K|L}} \mathrm{n}_{KL}, & \text{if } x \in \mathcal{D}_\sigma,\ \sigma = K|L \in \mathcal{E}_{\text{int}}; \\ 0, & \text{else .} \end{cases}$$

which is piecewise constant on the diamond cells $(\mathcal{D}_\sigma)_{\sigma \in \mathcal{E}_{int}}$.



Weak convergence and improved regularity

There exists a function $u \in L^2(\Omega; L^2(0, T; L^2(\Lambda)))$ such that, passing to a subsequence if necessary,

$$u^l_{h,N} \text{ and } u^r_{h,N} \rightharpoonup u$$

weakly in $L^2(\Omega; L^2(0, T; L^2(\Lambda)))$ for $N \to +\infty$, $h \to 0$.

Proposition [Eymard, Gallouët '03]

$u \in L^2(\Omega; L^2(0, T; H^1(\Lambda)))$ and

$$\nabla^h u^r_{h,N} \rightharpoonup \nabla u$$

weakly in $L^2(\Omega; L^2(0, T; L^2(\Lambda)^2)))$ for $N \to +\infty$, $h \to 0$

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weakly in $L^2(\Omega; L^2(0, T; L^2(\Lambda)^2)))$ for $N \to +\infty$, $h \to 0$

...but weak convergence is not compatible with the nonlinear diffusion term $g : \mathbb{R} \to \mathbb{R}$

Time and space translate estimates

Lemma [Bauzet, Nabet, Schmitz, Z. '22]

For any $\alpha \in (0, \frac{1}{2})$, $(u^l_{h,N})_{h,N}$ is bounded in the space

$$L^2(\Omega; L^2(0,T; W^{\alpha,2}(\Lambda))) \cap L^2(\Omega; W^{\alpha,2}(0,T; L^2(\Lambda))).$$

Idea of proof: Uniform estimates on the time and space translates of approximate solutions associated with (FV) are useful to find bounds on the Gagliardo seminorms for $(u^l_{h,N})_{h,N}$.

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$$L^2(0,T; W^{\alpha,2}(\Lambda)) \cap W^{\alpha,2}(0,T; L^2(\Lambda)) \overset{\text{compact}}{\hookrightarrow} L^2(0,T; L^2(\Lambda))$$

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$\overset{\text{Lemma}}{\Longrightarrow}$ The sequence of laws $\mathcal{L}(u^l_{h,N})_{h,N}$ on $L^2(0,T; L^2(\Lambda))$ is tight.

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$\overset{\text{Prokhorov}}{\Longrightarrow}$ Up to a subsequence, $(u^l_{h,N})_{h,N}$ converges in law to a probability measure μ_∞.

Theorem of Skorokhod

On a new probability space $(\Omega', \mathcal{A}', \mathbb{P}')$

- there exist random variables v_0, $(v^l_{h,N})_{h,N}$, u_∞ with

$$\mathcal{L}(v_0) = \mathcal{L}(u_0),\ \mathcal{L}(v^l_{h,N}) = \mathcal{L}(u^l_{h,N}) \text{ for all } h > 0,\ N \in \mathbb{N},$$

$$\mathcal{L}(u_\infty) = \mu_\infty$$

  and

$$v^l_{h,N} \xrightarrow{h,N} u_\infty \text{ in } L^2(0,T;L^2(\Lambda))\ \mathbb{P}'\text{-a.s. in } \Omega',$$

- there exists a stochastic process W^∞ and a sequence of Brownian motions $(W^{h,N})_{h,N}$ such that

$$W^{h,N} \xrightarrow{h,N} W^\infty \text{ in } \mathcal{C}([0,T])\ \mathbb{P}'\text{-a.s. in } \Omega'$$

Consequences of Skorokhod's Theorem

- $v^l_{h,N}$ is a step function, i.e., for any $K \in \mathcal{T}_h$, $n \in \{0, \dots, N-1\}$, $v^l_{h,N}(t,x) := v^n_K$ for all $t \in [t_n, t_{n+1})$, $x \in K$.
- For any $n \in \{0, \dots, N-1\}$, the random vector $(v^{n+1}_K)_{K \in \mathcal{T}_h}$ is a solution of

$$m_K(v^{n+1}_K - v^n_K) + \triangle t \sum_{\sigma \in \mathcal{E}_{\text{int}} \cap \mathcal{E}_K} \frac{m_\sigma}{d_{K|L}}(v^{n+1}_K - v^{n+1}_L) = m_K g(v^n_K)(W^{h,N}_{t_{n+1}} - W^{h,N}_{t_n}) \qquad \text{(FV')}$$

for all $K \in \mathcal{T}_h$.

Identification of the stochastic integral

- For any, $h > 0$, $N \in \mathbb{N}$ there exists a filtration $(\mathfrak{F}_t^{h,N})_{t\geq 0}$ such that $v_{h,N}^l$ is adapted to $(\mathfrak{F}_t^{h,N})_{t\geq 0}$ and $W^{h,N} = (W_t^{h,N})_{t\geq 0}$ is a Brownian motion with respect to $(\mathfrak{F}_t^{h,N})_{t\geq 0}$.

$$m_K(v_K^{n+1} - v_K^n) + \triangle t \sum_{\sigma\in\mathcal{E}_{\text{int}}\cap\mathcal{E}_K} \frac{m_\sigma}{d_{K|L}}(v_K^{n+1} - v_L^{n+1})$$
$$= m_K g(v_K^n)(W_{t_{n+1}}^{h,N} - W_{t_n}^{h,N})$$

for all $K \in \mathcal{T}_h$.

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$$
\begin{aligned}
& m_K(v_K^{n+1} - v_K^n) + \triangle t \sum_{\sigma\in\mathcal{E}_{\mathrm{int}}\cap\mathcal{E}_K} \frac{m_\sigma}{d_{K|L}}(v_K^{n+1} - v_L^{n+1}) \\
&= \int_K \int_{t_n}^{t_{n+1}} g(v_{h,N}^l)\, dW_t^{h,N}\, dx
\end{aligned}
$$

for all $K \in \mathcal{T}_h$.

Identification of the stochastic integral

- For any, $h > 0$, $N \in \mathbb{N}$ there exists a filtration $(\mathfrak{F}_t^{h,N})_{t\geq 0}$ such that $v_{h,N}^l$ is adapted to $(\mathfrak{F}_t^{h,N})_{t\geq 0}$ and $W^{h,N} = (W_t^{h,N})_{t\geq 0}$ is a Brownian motion with respect to $(\mathfrak{F}_t^{h,N})_{t\geq 0}$.
- There exists a filtration $(\mathfrak{F}_t^{\infty})_{t\geq 0}$ such that u_∞ has a predictable $d\mathbb{P}' \otimes dt$-representative and $W^{\infty} = (W_t^{\infty})_{t\geq 0}$ is a Brownian motion with respect to $(\mathfrak{F}_t^{\infty})_{t\geq 0}$.
- By a result of [Debussche, Glatt-Holtz, Temam 2011],

$$\int_\Lambda \int_0^T g(v_{h,N}^l)\, dW_t^{h,N}\, dx \xrightarrow{h,N} \int_\Lambda \int_0^T g(u_\infty)\, dW_t^{\infty}\, dx$$

$\mathbb{P}'$-a.s. in Ω'.

Strong convergence of finite-volume approximations

Proposition [Bauzet, Nabet, Schmitz, Z., '22]

$(\Omega', \mathbb{P}', \mathcal{A}', (\mathfrak{F}_t^\infty)_{t\geq 0}, W^\infty, u_\infty, v_0)$ is a martingale solution for the heat equation with multiplicative Lipschitz noise.

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- According to [Gyöngy, Krylov 1996]: up to a not relabeled subsequence, $(u^l_{h,N})_{h,N}$ and $(u^r_{h,N})_{h,N}$ converge $\mathbb{P}$-a.s. in $L^2(0,T;L^2(\Lambda))$

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- The joint limit u is the unique variational solution of the heat equation with multiplicative Lipschitz noise.

The stochastic heat equation with convection

$$\begin{aligned} du - \Delta u\, dt + \operatorname{div}(\mathbf{v}u)\, dt &= g(u)\, dW_t + \beta(u)\, dt && \text{in } \Omega \times (0,T) \times \Lambda \\ u(0,\cdot) &= u_0 && \text{in } \Omega \times \Lambda \\ \nabla u \cdot \mathrm{n} &= 0 && \text{on } \partial\Lambda \end{aligned}$$

- $\mathbf{v} \in C^1([0,T] \times \overline{\Lambda}; \mathbb{R}^2)$
- $\operatorname{div}(\mathbf{v}) = 0$ in $[0,T] \times \Lambda$
- $\mathbf{v} \cdot \vec{n} = 0$ on $[0,T] \times \partial\Lambda$
- $\beta : \mathbb{R} \to \mathbb{R}$ Lipschitz continuous with $\beta(0) = 0$

Upwind scheme for the convection term

For $u \in C^1(\overline{\Lambda}; \mathbb{R}), \mathbf{v} \in C^1(\overline{\Lambda}; \mathbb{R}^2)$ with $\mathbf{v} \cdot \vec{n} = 0$ on $\partial\Lambda$ and $K \in \mathcal{T}$

$$\begin{aligned}\int_K \operatorname{div}(\mathbf{v}(x)u(x))\, dx &= \int_{\partial K} \mathbf{v}(x) \cdot \vec{n}_K(x)u(x)\, d\gamma(x) \\ &= \sum_{\sigma \in \mathcal{E}_{\text{int}} \cap \mathcal{E}_K} \int_\sigma \mathbf{v}(x) \cdot \vec{n}_{K|L} u(x)\, d\gamma(x)\end{aligned}$$

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$$\begin{aligned}\int_K \operatorname{div}(\mathbf{v}(x)u(x))\,dx &= \int_{\partial K} \mathbf{v}(x) \cdot \vec{n}_K(x)u(x)\,d\gamma(x)\\ &= \sum_{\sigma\in\mathcal{E}_{\text{int}}\cap\mathcal{E}_K} \int_\sigma \mathbf{v}(x) \cdot \vec{n}_{K|L}u(x)\,d\gamma(x)\\ &\approx \sum_{\sigma\in\mathcal{E}_{\text{int}}\cap\mathcal{E}_K} m_\sigma v_{K,\sigma} u_\sigma.\end{aligned}$$

Where, u_σ is interpreted as the quantity of u transported through the interface $\sigma = K|L$ by the velocity $v_{K,\sigma}$:

$$v_{K,\sigma} := \frac{1}{m_\sigma}\int_\sigma \mathbf{v}(x) \cdot \vec{n}_{K|L}\,d\gamma(x), \quad u_\sigma := \begin{cases} u_K, & \text{if } v_{K,\sigma} \geq 0 \\ u_L, & \text{if } v_{K,\sigma} < 0. \end{cases}$$

Calculus for the upwind approximation

If $\operatorname{div}(\mathbf{v}) = 0$ in Λ, for $K \in \mathcal{T}$ we have

$$\sum_{\sigma \in \mathcal{E}_{\text{int}} \cap \mathcal{E}_K} m_\sigma v_{K,\sigma} u_\sigma = \sum_{\sigma \in \mathcal{E}_{\text{int}} \cap \mathcal{E}_K} m_\sigma v_{K,\sigma} (u_\sigma - u_K).$$

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since, for any $K \in \mathcal{T}$,

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\sum_{\sigma \in \mathcal{E}_{\text{int}} \cap \mathcal{E}_K} m_\sigma v_{K,\sigma} u_K &= \sum_{\sigma \in \mathcal{E}_{\text{int}} \cap \mathcal{E}_K} u_K \int_\sigma \mathbf{v}(x) \cdot \vec{n}_{K|L} \, d\gamma(x) \\
&= u_K \int_{\partial K} \mathbf{v}(x) \cdot \vec{n}_K(x) \, d\gamma(x) \\
&= u_K \int_K \operatorname{div}(\mathbf{v}(x)) \, dx = 0.
\end{aligned}$$

Moreover, since $v_{K,\sigma} = (v_{K,\sigma})^+ - (v_{K,\sigma})^-$, we have

$$\sum_{\sigma \in \mathcal{E}_{\text{int}} \cap \mathcal{E}_K} m_\sigma v_{K,\sigma} (u_\sigma - u_K) = \sum_{\sigma = K|L \in \mathcal{E}_{\text{int}} \cap \mathcal{E}_K} m_\sigma (v_{K,\sigma})^- (u_K - u_L).$$

Semi-implicit finite-volume scheme

Proposition [Bauzet, Nabet, Schmitz, Z., '23]

For any given $\mathcal{F}_{t_n}$-measurable random vector $(u_K^n)_{K\in\mathcal{T}}$ there exists a unique $\mathcal{F}_{t_{n+1}}$-measurable random vector $(u_K^{n+1})_{K\in\mathcal{T}}$ satisfying

$$\frac{m_K}{\Delta t}(u_K^{n+1}-u_K^n)+\sum_{\sigma\in\mathcal{E}_{\text{int}}\cap\mathcal{E}_K} m_\sigma (v_{K,\sigma}^{n+1})^-(u_K^{n+1}-u_L^{n+1})$$
$$+\sum_{\sigma\in\mathcal{E}_{\text{int}}\cap\mathcal{E}_K}\frac{m_\sigma}{d_{K|L}}(u_K^{n+1}-u_L^{n+1})$$
$$=\frac{m_K}{\Delta t}g(u_K^n)(W(t_{n+1})-W(t_n))+m_K\beta(u_K^{n+1}),$$

for all $K\in\mathcal{T}$, $\mathbb{P}$-a.s. in Ω, where, for $\sigma=K|L$

$$v_{K,\sigma}^{n+1}:=\frac{1}{m_\sigma\Delta t}\int_{t_n}^{t_{n+1}}\int_\sigma \mathrm{v}(t,x)\cdot \mathrm{n}_{KL}\,d\gamma(x)\,dt.$$

Convergence result

Theorem [Bauzet, Schmitz, Z., '23]

The finite-volume approximations $u^l_{h,N}$ and $u^r_{h,N}$ converge strongly in $L^2(\Omega; L^2(0,T; L^2(\Lambda)))$ to the unique solution of the stochastic heat equation with convection, i.e., to a predictable stochastic process $u \in L^2(\Omega; C([0,T]; L^2(\Lambda))) \cap L^2(\Omega; L^2(0,T; H^1(\Lambda)))$ such that

$$u(t) - u_0 - \int_0^t \Delta u(s)\, ds + \int_0^t \operatorname{div}(\mathbf{v}(s,\cdot)u(s))\, ds = \int_0^t g(u(s))\, dW_s + \int_0^t \beta(u(s))\, ds.$$

for all $t \in [0,T]$, in $L^2(\Lambda)$, $\mathbb{P}$-a.s. in Ω.

Proof of the convergence result

- Weak convergences of $g(u^l_{h,N})$, $\beta(u^r_{h,N})$ towards g_u and β_u, respectively
- Identification of the weak limits $g_u = g(u), \beta_u = \beta(u)$ via stochastic energy inequalities using an exponential weighted in time norm
- The key ingredient is

Lemma [Bauzet, Zimmermann, S. '23]

For any $c > 0$, the stochastic process u satisfies

$$\int_0^T \int_0^t e^{-cs}\mathbb{E}\left[\int_\Lambda |\nabla u(s,x)|^2\,dx\right]\,ds\,dt \leq \liminf_{h\to 0, N\to\infty} \int_0^T \int_0^t e^{-cs}\mathbb{E}\left[|u^r_{h,N}(s)|^2_{1,h}\right]\,ds\,dt.$$

where $|\cdot|_{1,h}$ is the discrete H^1-seminorm.

Thank you for your attention.