Well-posedness and Lewy-Stampaccia's inequalities for nonlinear stochastic evolution equations

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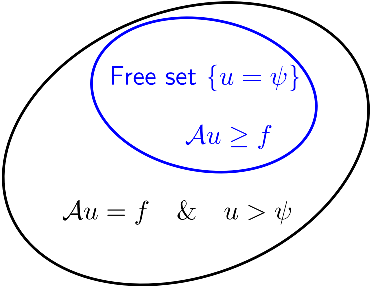
Stochastic Models in Mechanics: Theoretical and Numerical Aspects

Marseille, August 31st, 2023

Obstacle problems

General form: $\mathcal{A}u \geq f,\ u \geq \psi$ and $\langle \mathcal{A}u - f, u - \psi \rangle_{\mathcal{V}',\mathcal{V}} = 0$



- for elliptic PDE: $\mathcal{A}u = Au$ with $A : \mathcal{V} \to \mathcal{V}'$
- for parabolic PDE:

$$\mathcal{A}u = \partial_t u + Au$$

- for stochastic PDE:

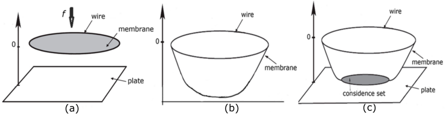
$$\mathcal{A}u = \partial_t \left(u - \int_0^{\cdot} G(u)\, dW \right) + Au$$

Lewy-Stampaccia's inequality: $0 \leq \mathcal{A}u - f \leq (f - \mathcal{A}\psi)^-$

where $r^- = -\min(0, r)$ for all $r \in \mathbb{R}$

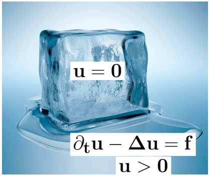
Obstacle problems in applications

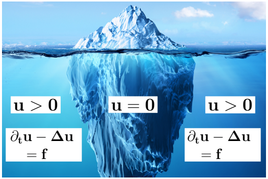
Signorini type problems



Apply a force on membrane | **without plate (no obstacle)** | **with plate (obstacle)**

Stefan type problems





Also: fluid flow in porous medium with a constraint on the pressure,
Model with constraints for vehicular traffic jams....

A stochastic pseudomonotone parabolic obstacle problem

Let $T > 0$, $D \subset \mathbb{R}^d$ a bounded Lipschitz domain, $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ a stochastic basis with the usual assumptions, $\Omega_T := \Omega \times (0, T)$. We study existence and uniqueness of solutions (u, ρ) to

$$
(\textbf{VI}) \begin{cases} du - \operatorname{div} a(u, \nabla u)\, dt + \rho\, dt = f\, dt + G(u)\, dW(t) & \text{in } D \times \Omega_T, \\ u(t, 0) = u_0 & \text{in } L^2(\Omega; L^2(D)) \\ u = 0 & \text{in } \partial D \times \Omega_T \\ u \geq \psi & \text{in } D \times \Omega_T \\ -\rho \geq 0 \text{ and } \langle \rho, u - \psi \rangle = 0 & \end{cases}
$$

- u_0 is $\mathcal{F}_0$-measurable, $u_0 \geq \psi(0)$
- $-\operatorname{div}\, a(v, \nabla v)$ is a pseudomonotone Leray-Lions operator from $W_0^{1,p}(D)$ to its dual space $W^{-1,p'}(D)$, $\max(1, \frac{2d}{d+2}) < p < \infty$
- $f \in L^{p'}(\Omega_T; W^{-1,p'}(D))$ is predictable
- ψ is an appropriate random obstacle function

Assumptions on the nonlinear operator

- $a : D \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ is a Carathéodory function on $D \times \mathbb{R}^{d+1}$,
- a is monotone, *i.e.*, for a.e. $x \in D$, for all $\lambda \in \mathbb{R}$, $\xi, \eta \in \mathbb{R}^d$,

$$(a(x, \lambda, \xi) - a(x, \lambda, \eta)) \cdot (\xi - \eta) \geq 0$$

- There exist $\bar{\alpha} > 0$ and $C_1^a, C_2^a, \bar{\gamma} \geq 0$, $\bar{h} \in L^1(D)$, $\bar{k} \in L^{p'}(D)$, and an exponent $q < p$ such that,

$$a(x, \lambda, \xi) \cdot \xi \geq \bar{\alpha}|\xi|^p - \bar{\gamma}|\lambda|^q + \bar{h}(x)$$
$$|a(x, \lambda, \xi)| \leq \bar{k}(x) + C_1^a|\lambda|^{p-1} + C_2^a|\xi|^{p-1}.$$

for a.e. $x \in D$, for all $\lambda \in \mathbb{R}$ and for all $\xi \in \mathbb{R}^d$, There exists $C_3^a \geq 0$ and a non-negative function $l \in L^{p'}(D)$ such that

$$|a(x, \lambda_1, \xi) - a(x, \lambda_2, \xi)| \leq \left(C_3^a|\xi|^{p-1} + l(x)\right)|\lambda_1 - \lambda_2|$$

for all λ_1, $\lambda_2 \in \mathbb{R}$, for all $\xi \in \mathbb{R}^d$ and a.e. $x \in D$.

Well-known example:
$-\operatorname{div} a(u, \nabla u) = -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u + F(u)\right)$, $p \geq 2$

Q-Wiener Process

- Let $(\beta^k)_k$ be a sequence of independent $(\mathcal{F}_t)_{t\geq 0}$-Brownian motions on $(\Omega, \mathcal{F}, \mathbb{P})$, i.e., for any $k \in \mathbb{N}$,
  - $\beta^k = (\beta^k(t))_{t\geq 0}$ is a real-valued stochastic process with $\beta^k(0) = 0$
  - For all $t \geq 0$, $\omega \mapsto \beta^k(t)(\omega)$ is a $\mathcal{F}_t$-measurable random variable
  - For all $0 \leq s \leq t$, the increments $\beta^k(t) - \beta^k(s)$ are $N(0, t-s)$ and independent of $\mathcal{F}_s$
- We fix a separable Hilbert space U such that $L^2(D) \subset U$ and a non-negative, symmetric trace class operator $Q : U \to U$ with

$$Q^{1/2}(U) = L^2(D)$$

- Let $(e_k)_k$ be an orthonormal basis of U made of eigenvectors of Q with corresponding eigenvalues $(\lambda_k)_k \subset [0, \infty)$.

Then,

$$W(t) := \sum_{k=1}^{\infty} \sqrt{\lambda_k} e_k \beta^k(t),\ t \geq 0$$

is a $(\mathcal{F}_t)_{t\geq 0}$-adapted Q-Wiener process with values in U.

The Itô integral for a Q-Wiener process

A linear operator $A : L^2(D) \to L^2(D)$ is a Hilbert-Schmidt operator, i.e., $A \in \mathrm{HS}(L^2(D))$, iff

$$\|A\|_{\mathrm{HS}}^2 := \sum_{k=1}^{\infty} \|A(Q^{1/2}(e_k))\|_{L^2(D)}^2 < \infty.$$

For a *predictable*, square-integrable process $\Phi : \Omega \times [0,T] \to \mathrm{HS}(L^2(D))$ and $t \in [0,T]$, the stochastic integral in the sense of Itô is given by

$$\int_0^t \Phi(s)\, dW = \sum_{k=1}^{\infty} \int_0^t \Phi(s)(\sqrt{\lambda_k} e_k)\, d\beta^k(s).$$

The Itô isometry holds true, i.e.,

$$\mathbb{E} \left\| \int_0^t \Phi(s)\, dW \right\|_{L^2(D)}^2 = \mathbb{E} \int_0^t \|\Phi(s)\|_{\mathrm{HS}}^2\, ds.$$

Assumptions on the noise

- $(W(t))_{t\geq 0}$ is a Q-Wiener process with respect to $(\mathcal{F}_t)_{t\geq 0}$
- Let $\mathrm{HS}(L^2(D))$ denote the space of Hilbert-Schmidt operators from $L^2(D)$ to $L^2(D)$
- The noise may be *additive* and *multiplicative* in the following sense: $G : \Omega_T \times L^2(D) \to \mathrm{HS}(L^2(D))$ is given by

$$G(\omega, t, u) = g(\omega, t) + \sigma(\omega, t, u)$$

  with predictable mappings $g \in L^2(\Omega_T; \mathrm{HS}(L^2(D)))$ and $\sigma : \Omega_T \times L^2(D) \to \mathrm{HS}(L^2(D))$.
- Moreover, $\sigma(\omega, t, 0) = 0$ and

$$\|\sigma(\omega, t, u) - \sigma(\omega, t, v)\|_{\mathrm{HS}} \leq L\|u - v\|_{L^2(D)}$$

  for a constant $L \geq 0$.

Assumptions on the obstacle

$\psi \in L^p(\Omega_T; W_0^{1,p}(D)) \cap L^2(\Omega_T; L^2(D))$ is predictable with

$$\partial_t \left(\psi - \int_0^\cdot G(\psi)\, dW \right) \in L^{p'}(\Omega_T; W^{-1,p'}(D))$$

The equation

$$du - \operatorname{div} a(u, \nabla u)\, dt + \rho\, dt = f\, dt + G(u)\, dW(t)$$

ist coupled to the obstacle by

Ordered dual assumption: There exist nonnegative elements h^+, h^- of $L^{p'}(\Omega_T; W^{-1,p'}(D))$ such that

$$f - \partial_t \left(\psi - \int_0^\cdot G(\psi)\, dW \right) + \operatorname{div}\, a(\psi, \nabla \psi) = h^+ - h^-$$

the ordered dual assumption provides additional regularity on ρ and well-posedness for Lewy-Stampaccia's inequality

Notion of solution

(u,ρ) is a solution to Problem **(VI)** iff:

- $u \in L^2(\Omega; C([0,T]; L^2(D)))$ is a predictable process
- $u \in L^p(\Omega_T; W_0^{1,p}(D))$, $u(0,\cdot) = u_0$
- $u \geq \psi$
- $\rho \in L^{p'}(\Omega_T; W^{-1,p'}(D))$ with $-\rho \geq 0$ and

$$\mathbb{E}\left[\int_0^T \langle \rho, u - \psi \rangle \, dt\right] = 0$$

- For all $t \in [0,T]$, $\mathbb{P}$-a.s. in Ω we have

$$u(t) - u_0 + \int_0^t - \operatorname{div}\, a(u, \nabla u) + \rho \, ds = \int_0^t f \, ds + \int_0^t G(u) \, dW$$

Lewy-Stampaccia's inequality

With the ordered dual assumption, a solution (u, ρ) to the variational inequality

$$du - \text{div } a(u, \nabla u)\, dt + \rho\, dt = f\, dt + G(u)\, dW(t),\ u \geq \psi,\ -\rho \geq 0$$

satisfies Lewy-Stampacchia's inequality iff

$$\begin{aligned} &\partial_t \left(u - \int_0^\cdot G(u)\, dW \right) - \text{div } a(u, \nabla u) - f \\ &\leq \left(f - \partial_t \left(\psi - \int_0^\cdot G(\psi)\, dW \right) + \text{div } a(\psi, \nabla \psi) \right)^- = h^- \end{aligned}$$

Main result

Theorem [STVZ; 2023]: There exists a unique solution (u,ρ) to **(VI)**.
Moreover, (u,ρ) satisfies Lewy-Stampacchia's inequality

$$\partial_t \left(u - \int_0^{\cdot} G(u)\, dW \right) - \mathrm{div}\; a(u, \nabla u) - f \leq h^-$$

Penalisation

For $\varepsilon > 0$, we consider the approximation

$$\partial_t \left(u_\varepsilon - \int_0^\cdot \widetilde{G}(u_\varepsilon)\, dW \right) - \operatorname{div} \widetilde{a}(u_\varepsilon, \nabla u_\varepsilon) - \frac{1}{\varepsilon}(u_\varepsilon - \psi)^- = f \qquad (P_\varepsilon)$$

where

- $\widetilde{G}(u_\varepsilon) = G(\omega, t, \max(\psi, u_\varepsilon)) = g(\omega, t) + \sigma(\omega, t, \max(\psi, u_\varepsilon))$
- $\widetilde{a}(u_\varepsilon, \nabla u_\varepsilon) = a(x, \max(\psi, u_\varepsilon), \nabla u_\varepsilon)$
- $(u_\varepsilon - \psi)^- = -\min(u_\varepsilon - \psi, 0)$

Existence and uniqueness of solutions u_ε to (P_ε) have to be provided.

A higher-order singular perturbation

- Let $\nu > \max\{p, 2, 2p(p-1)\}$ and $m \in \mathbb{N}$ be such that $H_0^m(D)$ has a continuous injection into $W_0^{1,2p}(D) \cap L^\infty(D)$
- $\partial J : W_0^{m,\nu}(D) \to W^{-m,\nu'}(D)$ is defined by

$$\langle \partial J(u), v \rangle = \sum_{|\alpha| \leq m} \int_D (1 + |D^\alpha u|^{\nu-2}) D^\alpha u D^\alpha v \, dx$$

Proposition [STVZ; 2023]: For $\varepsilon > 0$, $\delta > 0$ there exists a unique solution u_ε^δ to

$$du_\varepsilon^\delta + \left(\delta \partial J(u_\varepsilon^\delta) - \operatorname{div} \widetilde{a}(u_\varepsilon^\delta, \nabla u_\varepsilon^\delta) - \frac{1}{\varepsilon} \left(u_\varepsilon^\delta - \psi \right)^- \right) dt$$
$$= f \, dt + \widetilde{G}(u_\varepsilon^\delta) \, dW.$$

Proof: The conditions $(H1)$, $(H2')$, $(H3)$ and $(H4')$ of [Liu, Röckner; 2015, Section 5.1] are satisfied with $\beta = 0$ and $\alpha = \nu$

Compactness with respect to the parameter $\delta > 0$

For fixed $\varepsilon > 0$, the laws

$$\mu_\varepsilon^\delta = \mathcal{L}(u_\varepsilon^\delta, \widetilde{G}(u_\varepsilon^\delta), \psi, G(0), \sigma, W, f, u_0)$$

are tight. Prokhorov's theorem yields

$$\mu_\varepsilon^\delta \xrightarrow{\ *\ } \mu_\infty$$

up to a subsequence for $\delta \to 0^+$. Skorokhod's theorem yields the existence of random variables

$$(\overline{u}_\varepsilon^\delta, \overline{\widetilde{G}}(\overline{u}_\varepsilon^\delta), \overline{\psi}, \overline{G}_0, \overline{\sigma}, \overline{W}, \overline{f}, \overline{u_0})$$

on a new probability space $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}})$ such that

$$d\overline{u}_\varepsilon^\delta + \left(\delta \partial J(\overline{u}_\varepsilon^\delta) - \operatorname{div} \widetilde{a}(\overline{u}_\varepsilon^\delta, \nabla \overline{u}_\varepsilon^\delta) - \frac{1}{\varepsilon}(\overline{u}_\varepsilon^\delta - \overline{\psi})^-\right) dt$$
$$= \overline{f}\, dt + \overline{\widetilde{G}}(\overline{u}_\varepsilon^\delta)\, d\overline{W}$$

and $\overline{u}_\varepsilon^\delta$ converges to a random variable $\overline{u}_\varepsilon^\infty$ a.s. in $\overline{\Omega}$ for $\delta \to 0^+$

Passage to the limit for $\delta \to 0^+$

For $\varepsilon > 0$ fixed

- We use the $\overline{\mathbb{P}}$-a.s. convergence of $\overline{u}_\varepsilon^\delta$ to a random variable $\overline{u}_\varepsilon^\infty$ in $\overline{\Omega}$ given by Skorohkod's theorem, Vitali's theorem, and weak convergence results from the a-priori estimates to arrive at

$$d\overline{u}_\varepsilon^\infty + \left(A^\infty - \frac{1}{\varepsilon}(\overline{u}_\varepsilon^\infty - \overline{\psi})^- \right) dt = \overline{f}\, dt + \widetilde{\overline{G}}(\overline{u}_\varepsilon^\infty)\, d\overline{W}$$

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- From an energy inequality with respect to a weighted exponential norm and a stochastic version of Minty's trick adapted from [Roubiček; 2005, Lemma 8.8] we get

$$A^\infty = -\operatorname{div} \widetilde{a}(\overline{u}_\varepsilon^\infty, \nabla \overline{u}_\varepsilon^\infty)$$

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Passage to the limit for $\delta \to 0^+$

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$$du_\varepsilon + \left(-\operatorname{div}\, \widetilde{a}(u_\varepsilon, \nabla u_\varepsilon) - \frac{1}{\varepsilon}(u_\varepsilon - \psi)^-\right) dt = f\, dt + \widetilde{G}(u_\varepsilon)\, dW$$

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The variational inequality in the regular case

$$du_\varepsilon + \left(-\operatorname{div}\, \widetilde{a}(u_\varepsilon, \nabla u_\varepsilon) - \frac{1}{\varepsilon}(u_\varepsilon - \psi)^- \right) dt = f\, dt + \widetilde{G}(u_\varepsilon)\, dW$$

Ordered dual assumption with regularity: $h^- \in L^\alpha(\Omega_T; L^\alpha(D))$ for $\alpha = \max(2, p')$ and

$$f - \partial_t \left(\psi - \int_0^\cdot G(\psi)\, dW \right) + \operatorname{div}\, a(\psi, \nabla\psi) = h^+ - h^-$$

- Thanks to the penalisation term $-\frac{1}{\varepsilon}(u_\varepsilon - \psi)^-$, the sequence $(u_\varepsilon)_{\varepsilon \geq 0}$ is nondecreasing for $\varepsilon \to 0^+$,
- Thanks to the regularity of h^-, $-\frac{1}{\varepsilon}(u_\varepsilon - \psi)^-$ is bounded in $L^\alpha(\Omega_T; L^\alpha(D))$ and $(u_\varepsilon - \psi)^- \to 0$ in $L^2(\Omega_T; L^2(D))$
- For $u := \sup_{\varepsilon > 0} u_\varepsilon, \quad \rho :=$ weak- $\lim_{\varepsilon \to 0^+} -\frac{1}{\varepsilon}(u_\varepsilon - \psi)^-$ we have

$$u \geq \psi, \ -\rho \geq 0 \text{ and } \mathbb{E}\left[\int_0^T \int_D \rho(u - \psi)\, dx\, dt = 0 \right]$$

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$$u \geq \psi, \; -\rho \geq 0 \text{ and } \mathbb{E}\left[\int_0^T \int_D \rho(u - \psi)\, dx\, dt = 0 \right]$$

Lewy-Stampaccia's inequalities in the regular case

$$h^- \in L^p(\Omega_T; W_0^{1,p}(D)) \cap L^\alpha(\Omega_T; L^\alpha(D)), \ \partial_t h^- \in L^2(\Omega_T; L^2(D))$$

One has to prove:

$$-\rho \leq h^-$$

- For $\varepsilon > 0$ it holds true that

$$-\frac{1}{\varepsilon}(u_\varepsilon - \psi)^- = -\frac{1}{\varepsilon}(u_\varepsilon - \psi)^- + h^- - h^-$$

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$$-\frac{1}{\varepsilon}(u_\varepsilon - \psi)^- \geq -\frac{1}{\varepsilon}(u_\varepsilon - \psi + \varepsilon h^-)^- - h^-$$

- We write the SPDE for $(u_\varepsilon - \psi + \varepsilon h^-)$

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- We write the SPDE for $(u_\varepsilon - \psi + \varepsilon h^-)$
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- We use Itô formula for a smooth approximation of $(r^-)^2$ to obtain the SPDE for $\|(u_\varepsilon - \psi + \varepsilon h^-)^-\|^2_{L^2(D)}$
- For $\varepsilon \to 0^+$ then it follows that

$$-\frac{1}{\varepsilon}(u_\varepsilon - \psi + \varepsilon h^-)^- \to 0 \text{ in } L^2(\Omega_T; L^2(D))$$

Lewy-Stampaccia's inequalities in the regular case

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One has to prove

$$-\rho \leq h^-$$

- For $\varepsilon > 0$ it holds true that

For $\varphi \geq 0$, $\varepsilon > 0$

$$\int_D -\frac{1}{\varepsilon}(u_\varepsilon - \psi)^- \varphi \, dx \geq \int_D (-\frac{1}{\varepsilon}(u_\varepsilon - \psi + \varepsilon h^-)^- - h^-)\varphi \, dx$$

- We write the SPDE for $(u_\varepsilon - \psi + \varepsilon h^-)$
- We use Itô formula for a smooth approximation of $(r^-)^2$ to obtain the SPDE for $\|(u_\varepsilon - \psi + \varepsilon h^-)^-\|^2_{L^2(D)}$
- For $\varepsilon \to 0^+$ then it follows that

$$-\frac{1}{\varepsilon}(u_\varepsilon - \psi + \varepsilon h^-)^- \to 0 \text{ in } L^2(\Omega_T; L^2(D))$$

Lewy-Stampaccia's inequalities in the regular case

$$h^- \in L^p(\Omega_T; W_0^{1,p}(D)) \cap L^\alpha(\Omega_T; L^\alpha(D)),\ \partial_t h^- \in L^2(\Omega_T; L^2(D))$$

One has to prove

$$-\rho \leq h^-$$

- For $\varepsilon > 0$ it holds true that

For $\varphi \geq 0$, $\varepsilon \to 0^+$

$$\int_D \rho\varphi \, dx \geq \int_D -h^- \varphi \, dx$$

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General ordered dual assumption

$$h^- \in L^{p'}(\Omega_T; W^{-1,p'}(D))$$

- There exists an approximating sequence of regular, non-negative mappings (h_n) such that

$$h_n \xrightarrow{n\to\infty} h^- \text{ in } L^{p'}(\Omega_T; W^{-1,p'}(D))$$

- Associated with h_n, for $n \in \mathbb{N}$ we define

$$f_n = \partial_t \left(\psi - \int_0^{\cdot} G(\psi)\, dW \right) - \operatorname{div}\, a(\cdot, \psi, \nabla\psi) + h^+ - h_n$$

- f_n converges to f in $L^{p'}(\Omega_T; W^{-1,p'}(D))$ for $n \to \infty$
- There exists a unique solution (u_n, ρ_n) to

$$du_n(t) + (-\operatorname{div}\, a(\cdot, u_n, \nabla u_n) + \rho_n)\, dt = f_n\, dt + G(u_n)\, dW$$

  such that $u_n \geq \psi$, $-\rho_n \geq 0$ and $-\rho_n \leq h_n$
- weak convergence of ρ_n to ρ for $n \to \infty$ yield $-\rho \leq h^-$
- Passage to the limit in the equation: Prokhorov's and Skorokhod's theorem is needed. Pathwise uniqueness is available at the limit.

Thank you for your attention