The Two-Point Flux Approximation scheme for parabolic problems

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Workshop Stochastic models in mechanics
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Principle

- Based on the conservation form of the PDE:

$$\mathrm{div}(\textbf{something}) = \text{source}.$$

- Integrate the balance equation on each cell $\mathcal{K}$ and apply Stokes formula:

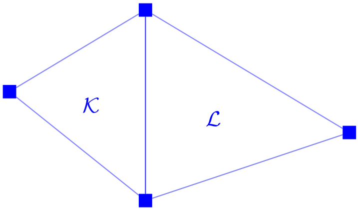
$$\begin{aligned}\int_{\mathcal{K}} \text{source} &= \int_{\partial\mathcal{K}} \text{Outward flux of } \textbf{something} \text{ across the edge} \\ &= \sum_{\text{edges of } \mathcal{K}} \text{Outward flux of } \textbf{something} \text{ across the edge.}\end{aligned}$$

- Approximate each flux and write the discrete balance equation obtained from this approximation.

Consider the following problem

$$\begin{cases} -\Delta u = f, & \text{in } \Lambda \\ \quad + \ \text{B.C.} & \text{on } \partial\Lambda \end{cases}$$

and an admissible Delaunay orthogonal mesh $\mathcal{T}$.



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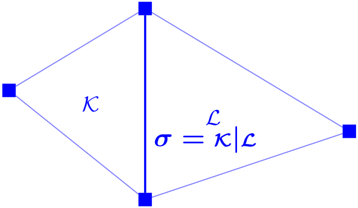

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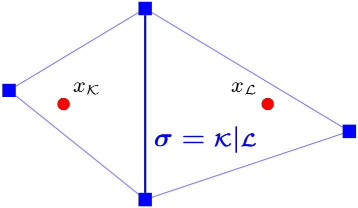

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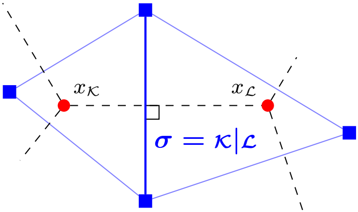

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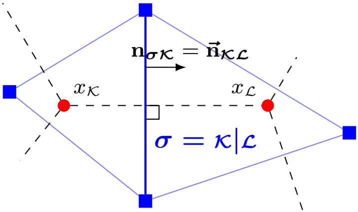

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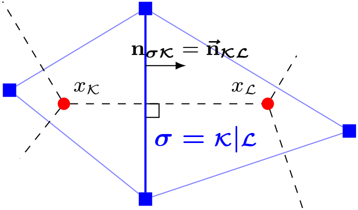
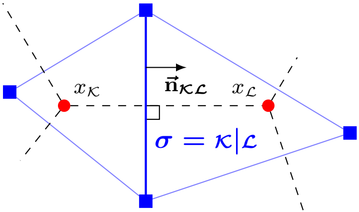


FLUX BALANCE EQUATION ON THE CONTROL VOLUME $\mathcal{K}$

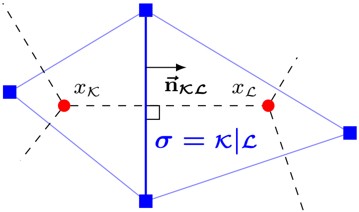
$$m_{\mathcal{K}} f_{\mathcal{K}} \stackrel{\text{def}}{=} \int_{\mathcal{K}} f = \int_{\mathcal{K}} -\Delta u = -\int_{\partial\mathcal{K}} \nabla u \cdot \mathbf{n}_{\sigma\mathcal{K}} = \sum_{\sigma\subset\partial\mathcal{K}} \underbrace{-\int_{\sigma} \nabla u \cdot \mathbf{n}_{\sigma\mathcal{K}}}_{\stackrel{\text{def}}{=} \overline{F}_{\mathcal{K},\sigma}(u)}$$



$$m_{\mathcal{K}} f_{\mathcal{K}} = \sum_{\sigma \subset \partial\mathcal{K}} \overline{F}_{\mathcal{K},\sigma}(u).$$

Local conservativity property for the initial problem

$$\overline{F}_{\mathcal{K},\sigma}(u) = -\overline{F}_{\mathcal{L},\sigma}(u), \quad \text{for } \sigma = \mathcal{K}|\mathcal{L}.$$



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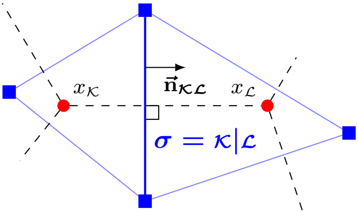
Cell-centered unknowns

We are looking for $u_\mathcal{K} \sim u(x_\mathcal{K})$

$$\textbf{Notation :} \; u_\mathcal{T} = (u_\mathcal{K})_{\mathcal{K}\in\mathcal{T}} \in \mathbb{R}^\mathcal{T}.$$



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Numerical fluxes

A family of maps $u_\mathcal{T} \mapsto F_{\mathcal{K},\sigma}(u_\mathcal{T})$ in order to approximate $\overline{F}_{\mathcal{K},\sigma}(u)$.

Numerical scheme

We look for $u_\mathcal{T} \in \mathbb{R}^\mathcal{T}$ such that $m_\mathcal{K} f_\mathcal{K} = \sum_{\sigma \subset \partial \mathcal{K}} F_{\mathcal{K},\sigma}(u_\mathcal{T})$ for any $\mathcal{K} \in \mathcal{T}$.

Case of an interior edge

$\sigma = \mathcal{K}|\mathcal{L}$.

$$u(x_{\mathcal{L}}) - u(x_{\mathcal{K}}) = (\nabla u(x)) \cdot (x_{\mathcal{L}} - x_{\mathcal{K}}) + \mathcal{O}\left(\text{size}(\mathcal{T})^2\right)$$

$$(x_{\mathcal{L}} - x_{\mathcal{K}}) = d_{\mathcal{K},\mathcal{L}} \vec{\mathbf{n}}_{\mathcal{KL}} \;\Rightarrow\; (\nabla u(x)) \cdot \vec{\mathbf{n}}_{\mathcal{KL}} = \frac{u(x_{\mathcal{L}}) - u(x_{\mathcal{K}})}{d_{\mathcal{K},\mathcal{L}}} + \mathcal{O}\left(\text{size}(\mathcal{T})\right)$$

$$\overline{F}_{\mathcal{K},\sigma}(u) = \int_{\sigma} -\nabla u \cdot \vec{\mathbf{n}}_{\mathcal{KL}} = -m_{\sigma} \frac{u(x_{\mathcal{L}}) - u(x_{\mathcal{K}})}{d_{\mathcal{K},\mathcal{L}}} + \mathcal{O}\left(\text{size}(\mathcal{T})^2\right)$$

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$$\boxed{F_{\mathcal{K},\sigma}(u_{\mathcal{T}}) \stackrel{\text{def}}{=} -m_{\sigma} \frac{u_{\mathcal{L}} - u_{\mathcal{K}}}{d_{\mathcal{K},\mathcal{L}}}.}$$

Remark and definition

The scheme is built so as to be conservative

$$F_{\mathcal{K},\sigma}(\mathcal{T}) = -F_{\mathcal{L},\sigma}(\mathcal{T}).$$

We set

$$F_{\mathcal{K},\mathcal{L}}(\mathcal{T}) \stackrel{\text{def}}{=} F_{\mathcal{K},\sigma}(\mathcal{T}) = -F_{\mathcal{L},\sigma}(\mathcal{T}).$$

Case of a boundary edge

$\sigma \subset \partial\Lambda.$

- Dirichlet B.C.: $u = 0$ on $\partial\Lambda$

$$x_\sigma - x_\mathcal{K} = d_{\mathcal{K},\sigma}\mathbf{n}_{\sigma\mathcal{K}}.$$

$$(\nabla u(x)) \cdot \mathbf{n}_{\sigma\mathcal{K}} \sim \frac{u(x_\sigma) - u(x_\mathcal{K})}{d_{\mathcal{K},\sigma}} = \frac{\mathbf{0} - u(x_\mathcal{K})}{d_{\mathcal{K},\sigma}} \Leftarrow \text{Boundary data}$$

$$\Longrightarrow \overline{F}_{\mathcal{K},\sigma}(u) = -m_\sigma \frac{-u(x_\mathcal{K})}{d_{\mathcal{K},\sigma}} + O(\text{size}(\mathcal{T})^2)$$

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- Neumann B.C.: $\nabla u \cdot \mathbf{n} = 0$ on $\partial\Lambda$
  By definition,

$$\overline{F}_{\mathcal{K},\sigma}(u) = \int_\sigma \nabla u \cdot \mathbf{n}_{\boldsymbol{\sigma}\boldsymbol{\mathcal{K}}} \quad \text{and} \quad (\nabla u(x)) \cdot \mathbf{n}_{\boldsymbol{\sigma}\boldsymbol{\mathcal{K}}} = 0, \quad \forall x \in \sigma.$$

$$F_{\mathcal{K},\sigma}(u_\mathcal{T}) \stackrel{\text{def}}{=} 0.$$

$\Rightarrow$ scheme only given on interior edges!

Definition of the TPFA scheme

We look for $u_{\mathcal{T}} = (u_{\mathcal{K}})_{\mathcal{K}\in\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}$ such that

$$
\text{(TPFA)} \quad \begin{cases}
\displaystyle\sum_{\sigma\subset\partial\mathcal{K}} F_{\mathcal{K},\sigma}(u_{\mathcal{T}}) = m_{\mathcal{K}} f_{\mathcal{K}}, & \forall \mathcal{K}\in\mathcal{T}, \\
F_{\mathcal{K},\sigma}(u_{\mathcal{T}}) = -m_\sigma \dfrac{u_{\mathcal{L}} - u_{\mathcal{K}}}{d_{\mathcal{K},\mathcal{L}}}, & \text{for } \sigma = \kappa|\mathcal{L}, \\
F_{\mathcal{K},\sigma}(u_{\mathcal{T}}) = -m_\sigma \dfrac{-u_{\mathcal{K}}}{d_{\mathcal{K},\sigma}}, & \text{for } \sigma\subset\partial\kappa \text{ with Dirichel B.C.}, \\
F_{\mathcal{K},\sigma}(u_{\mathcal{T}}) = 0, & \text{for } \sigma\subset\partial\kappa \text{ with Neumann B.C.}
\end{cases}
$$

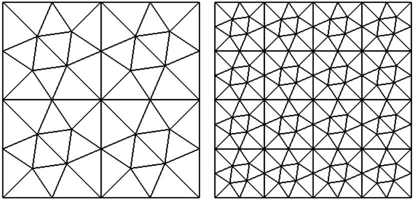
- It is a linear system of N equations with N unknowns (N=nb of control volumes in $\mathcal{T}$) which is invertible.
- The scheme is also known as VF4/FV4 : 4-point stencil for a triangle 2D mesh.
- On a 2D uniform Cartesian mesh : we recover the usual 5-point scheme.
- We can show that the scheme converges, we can obtain error estimates...

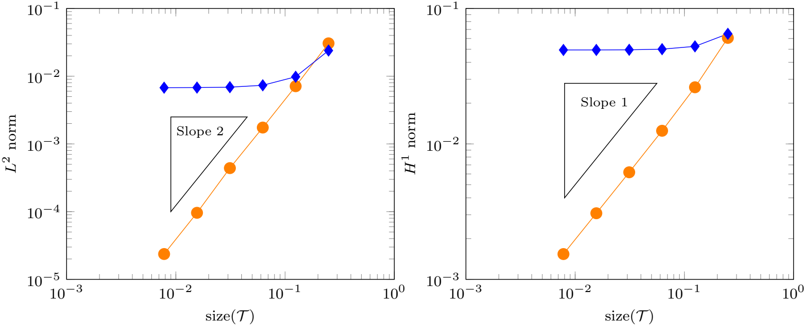
EXACT SOLUTION:

$u(x, y) = \sin(\pi x)\sin(\pi y)$

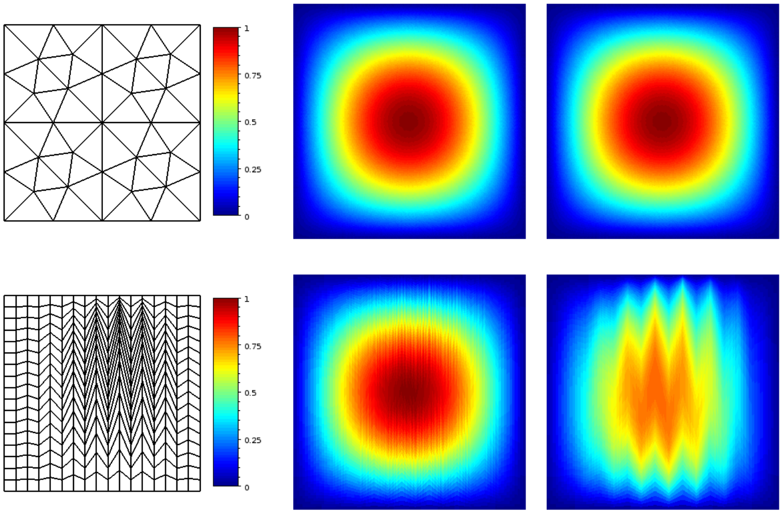
CENTERS:

- Circumcenter
- Center of mass





Error L^2 / H^1



Mesh

Exact solution

Approximate solution

A more complex problem

Find $u : (0,T) \times \Lambda \to \mathbb{R}$ s.t:

$$\begin{cases} \partial_t u - \Delta u + \operatorname{div}(\mathbf{v} f(u)) - \beta(u) = 0, & \text{in } (0,T) \times \Lambda; \\ \nabla u \cdot \mathbf{n} = 0, & \text{on } (0,T) \times \partial\Lambda; \\ u(0,\cdot) = u_0, & \text{in } \Lambda. \end{cases}$$

with

- $f : \mathbb{R} \to \mathbb{R}$ Lipschitz continuous, nondecreasing;
- $\beta : \mathbb{R} \to \mathbb{R}$ Lipschitz continuous with $\beta(0) = 0$.
- $\mathbf{v} \in \mathscr{C}^1([0,T] \times \overline{\Lambda}; \mathbb{R}^d)$ such that $\operatorname{div}(\mathbf{v}) = 0$ in $[0,T] \times \Lambda$ and $\mathbf{v} \cdot \mathbf{n} = 0$ on $[0,T] \times \partial\Lambda$.

A more complex problem

Find $u : (0,T) \times \Lambda \to \mathbb{R}$ s.t:

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Time discretization

$N \in \mathbb{N}^* \Rightarrow$ Time step: $\Delta t = \dfrac{T}{N}$ and $\forall n \in [\![0, N]\!]$, $t^n = n\Delta t$.

We are looking for $u_{\mathcal{K}}^n \sim u(t^n, x_{\mathcal{K}})$.

To obtain the scheme $$\partial_t u - \Delta u + \mathrm{div}(\mathbf{v} f(u)) - \beta(u) = 0$$

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- Integration

$$\int_{\mathcal{K}} \int_{t^n}^{t^{n+1}} \Big(\partial_t u - \Delta u + \mathrm{div}(\mathbf{v} f(u)) - \beta(u) \Big) \mathrm{d}t \mathrm{d}x = 0$$

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$$\int_{\mathcal{K}} \left(u(t^{n+1}, x) - u(t^n, x)\right) \mathrm{d}x - \int_{\mathcal{K}} \int_{t^n}^{t^{n+1}} \Delta u(t, x) \mathrm{d}t \mathrm{d}x + \int_{\mathcal{K}} \int_{t^n}^{t^{n+1}} \operatorname{div}(\mathbf{v}(t, x) f(u(t, x))) \mathrm{d}t \mathrm{d}x - \int_{\mathcal{K}} \int_{t^n}^{t^{n+1}} \beta(u(t, x)) \mathrm{d}x = 0.$$

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$$+ \int_{\mathcal{K}} \int_{t^n}^{t^{n+1}} \mathrm{div}(\mathbf{v}(t,x) f(u(t,x))) \mathrm{d}t \mathrm{d}x - \int_{\mathcal{K}} \int_{t^n}^{t^{n+1}} \beta(u(t,x)) \mathrm{d}x = 0.$$

- Implicit formulation

$$\frac{1}{\Delta t} \int_{\mathcal{K}} \left(u(t^{n+1},x) - u(t^n,x)\right) \mathrm{d}x - \int_{\mathcal{K}} \Delta u(t^{n+1},x) \mathrm{d}x$$
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- Stokes formula

$$\frac{1}{\Delta t} \int_{\mathcal{K}} \left(u(t^{n+1},x) - u(t^n,x)\right) \mathrm{d}x - \int_{\partial\mathcal{K}} \nabla u(t^{n+1},x) \cdot \mathbf{n}_{\boldsymbol{\sigma}\boldsymbol{\mathcal{K}}}(x) \mathrm{d}\gamma(x)$$
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$$\frac{1}{\Delta t}\int_{\mathcal{K}}\left(u(t^{n+1},x)-u(t^{n},x)\right)\mathrm{d}x-\sum_{\sigma\subset\partial\mathcal{K}}\int_{\sigma}\nabla u(t^{n+1},x)\cdot\mathbf{n}_{\boldsymbol{\sigma}\boldsymbol{\mathcal{K}}}(x)\mathrm{d}\gamma(x)$$
$$+\frac{1}{\Delta t}\sum_{\sigma\subset\partial\mathcal{K}}\int_{\sigma}f(u(t^{n+1},x))\int_{t^{n}}^{t^{n+1}}\mathbf{v}(t,x)\cdot\mathbf{n}_{\boldsymbol{\sigma}\boldsymbol{\mathcal{K}}}(x)\mathrm{d}t\mathrm{d}\gamma(x)-\int_{\mathcal{K}}\beta(u(t^{n+1},x))\mathrm{d}x=0.$$

The TPFA scheme

$$m_{\mathcal{K}}\frac{u_{\mathcal{K}}^{n+1}-u_{\mathcal{K}}^{n}}{\Delta t}+\sum_{\substack{\sigma\subset\partial\mathcal{K}\\ \sigma\notin\partial\Lambda}}\frac{m_{\sigma}}{d_{\mathcal{K},\mathcal{L}}}(u_{\mathcal{K}}^{n+1}-u_{\mathcal{L}}^{n+1})-m_{\mathcal{K}}\beta(u_{\mathcal{K}}^{n+1})$$
$$+\sum_{\substack{\sigma\subset\partial\mathcal{K}\\ \sigma\notin\partial\Lambda}}m_{\sigma}f(u_{\sigma}^{n+1})\underbrace{\frac{1}{m_{\sigma}\Delta t}\int_{\sigma}\int_{t^{n}}^{t^{n+1}}\mathbf{v}(t,x)\cdot\mathbf{n}_{\boldsymbol{\sigma}\boldsymbol{\mathcal{K}}}(x)\mathrm{d}t\mathrm{d}\gamma(x)}_{:=v_{\mathcal{K},\sigma}^{n+1}}=0.$$

THE TPFA SCHEME

$$m_{\mathcal{K}} \frac{u_{\mathcal{K}}^{n+1} - u_{\mathcal{K}}^{n}}{\Delta t} + \sum_{\substack{\sigma \subset \partial \mathcal{K} \\ \sigma \notin \partial \Lambda}} \frac{m_\sigma}{d_{\mathcal{K},\mathcal{L}}} (u_{\mathcal{K}}^{n+1} - u_{\mathcal{L}}^{n+1}) - m_{\mathcal{K}} \beta(u_{\mathcal{K}}^{n+1})$$
$$+ \sum_{\substack{\sigma \subset \partial \mathcal{K} \\ \sigma \notin \partial \Lambda}} m_\sigma f(u_\sigma^{n+1}) \underbrace{\frac{1}{m_\sigma \Delta t} \int_\sigma \int_{t^n}^{t^{n+1}} \mathbf{v}(t,x) \cdot \mathbf{n}_{\boldsymbol{\sigma}\boldsymbol{\mathcal{K}}}(x) \mathrm{d}t \mathrm{d}\gamma(x)}_{:= v_{\mathcal{K},\sigma}^{n+1}} = 0.$$

CHOICE OF u_σ^{n+1}

$$u_\sigma^{n+1} = \begin{cases} u_{\mathcal{K}}^{n+1} \text{ if } v_{\mathcal{K},\sigma}^{n+1} \geq 0, \\ u_{\mathcal{L}}^{n+1} \text{ if } v_{\mathcal{K},\sigma}^{n+1} < 0. \end{cases}$$

The TPFA scheme

$$m_{\mathcal{K}} \frac{u_{\mathcal{K}}^{n+1} - u_{\mathcal{K}}^{n}}{\Delta t} + \sum_{\substack{\sigma \subset \partial \mathcal{K} \\ \sigma \notin \partial \Lambda}} \frac{m_\sigma}{d_{\mathcal{K},\mathcal{L}}} (u_{\mathcal{K}}^{n+1} - u_{\mathcal{L}}^{n+1}) - m_{\mathcal{K}} \beta(u_{\mathcal{K}}^{n+1})$$

$$+ \sum_{\substack{\sigma \subset \partial \mathcal{K} \\ \sigma \notin \partial \Lambda}} m_\sigma f(u_\sigma^{n+1}) \underbrace{\frac{1}{m_\sigma \Delta t} \int_\sigma \int_{t^n}^{t^{n+1}} \mathbf{v}(t,x) \cdot \mathbf{n}_{\boldsymbol{\sigma}\boldsymbol{\mathcal{K}}}(x) \mathrm{d}t \mathrm{d}\gamma(x)}_{:= v_{\mathcal{K},\sigma}^{n+1}} = 0.$$

Choice of u_σ^{n+1}

$$u_\sigma^{n+1} = \begin{cases} u_{\mathcal{K}}^{n+1} \text{ if } v_{\mathcal{K},\sigma}^{n+1} \geq 0, \\ u_{\mathcal{L}}^{n+1} \text{ if } v_{\mathcal{K},\sigma}^{n+1} < 0. \end{cases}$$

The numerical scheme

Let $u_{\mathcal{T}}^n \in \mathbb{R}^{\mathcal{T}}$ be given. We look for $u_{\mathcal{T}}^{n+1} \in \mathbb{R}^{\mathcal{T}}$ s.t. for any $\mathcal{K} \in \mathcal{T}$,

$$m_{\mathcal{K}} \frac{u_{\mathcal{K}}^{n+1} - u_{\mathcal{K}}^{n}}{\Delta t} + \sum_{\substack{\sigma \subset \partial \mathcal{K} \\ \sigma \notin \partial \Lambda}} \frac{m_\sigma}{d_{\mathcal{K},\mathcal{L}}} (u_{\mathcal{K}}^{n+1} - u_{\mathcal{L}}^{n+1}) - m_{\mathcal{K}} \beta(u_{\mathcal{K}}^{n+1})$$

$$+ \sum_{\substack{\sigma \subset \partial \mathcal{K} \\ \sigma \notin \partial \Lambda}} m_\sigma v_{\mathcal{K},\sigma}^{n+1} f(u_\sigma^{n+1}) = 0.$$

PIECEWISE CONSTANT APPROXIMATION

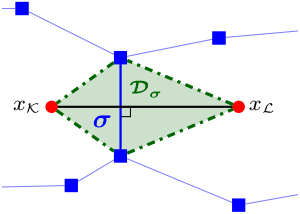
With each set of unknowns $v_{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}$, we associate the **piecewise constant** function

$$v_{\mathcal{T}}(x) = \sum_{\mathcal{K} \in \mathcal{T}} v_{\mathcal{K}} \mathbf{1}_{\mathcal{K}}(x).$$

DISCRETE L^2-NORM AND H^1-SEMINORM

$$\|v_{\mathcal{T}}\|^2_{L^2(\Lambda)} = \sum_{\mathcal{K} \in \mathcal{T}} m_{\mathcal{K}} |v_{\mathcal{K}}|^2 \quad \text{and} \quad |v_{\mathcal{T}}|^2_{1,\mathcal{T}} = \sum_{\sigma \in \mathcal{E}^{\text{int}}} \frac{m_\sigma}{d_{\mathcal{K},\mathcal{L}}} (v_{\mathcal{K}} - v_{\mathcal{L}})^2.$$

DIAMOND CELLS AND DISCRETE GRADIENT



$\forall v_{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}$, $\forall \sigma \in \mathcal{E}^{\text{int}}$, we set

$$\nabla^{\mathcal{T}}_{\mathcal{D}} v_{\mathcal{T}} \stackrel{\text{def}}{=} d \frac{v_{\mathcal{L}} - v_{\mathcal{K}}}{d_{\mathcal{K},\mathcal{L}}} \vec{\mathbf{n}}_{\mathcal{K}\mathcal{L}},$$

$$\nabla^{\mathcal{T}} v_{\mathcal{T}} \stackrel{\text{def}}{=} \sum_{\sigma \in \mathcal{E}^{\text{int}}} \mathbf{1}_{\mathcal{D}} \nabla^{\mathcal{T}}_{\mathcal{D}} v_{\mathcal{T}} \in (L^2(\Lambda))^2.$$

LINK WITH THE DISCRETE H^1 SEMI-NORM

$$|v_{\mathcal{T}}|^2_{1,\mathcal{T}} = \frac{1}{d} \left\| \nabla^{\mathcal{T}} v_{\mathcal{T}} \right\|^2_{L^2(\Lambda)}.$$

Proposition (Bounds on the solution)

Let $u_{\mathcal{T}}^n \in \mathbb{R}^{\mathcal{T}}$ be given. We assume there exists a solution $u_{\mathcal{T}}^{n+1}$ to the scheme, then there exists $M > 0$ independent of $\text{size}(\mathcal{T})$ *and* Δt *s.t.:*

$$\sup_{n \leq N} \|u_{\mathcal{T}}^n\|_{L^2(\Lambda)} \leq M \quad \textit{and} \quad \sum_{n=0}^{N-1} \Delta t \left|u_{\mathcal{T}}^{n+1}\right|_{1,\mathcal{T}}^2 \leq M.$$

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Proof:

- Scheme

$$m_{\mathcal{K}}(u_{\mathcal{K}}^{n+1} - u_{\mathcal{K}}^n) + \Delta t \sum_{\substack{\sigma \subset \partial\mathcal{K} \\ \sigma \notin \partial\Lambda}} \frac{m_\sigma}{d_{\mathcal{K},\mathcal{L}}}(u_{\mathcal{K}}^{n+1} - u_{\mathcal{L}}^{n+1}) + \Delta t \sum_{\substack{\sigma \subset \partial\mathcal{K} \\ \sigma \notin \partial\Lambda}} m_\sigma v_{\mathcal{K},\sigma}^{n+1} f(u_\sigma^{n+1}) = m_{\mathcal{K}} \beta(u_{\mathcal{K}}^{n+1})$$

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Proof:

- Scheme

$$\sum_{\mathcal{K} \in \mathcal{T}} u_{\mathcal{K}}^{n+1} \Bigg(m_{\mathcal{K}} (u_{\mathcal{K}}^{n+1} - u_{\mathcal{K}}^n) + \Delta t \sum_{\substack{\sigma \subset \partial \mathcal{K} \\ \sigma \notin \partial \Lambda}} \frac{m_\sigma}{d_{\mathcal{K},\mathcal{L}}} (u_{\mathcal{K}}^{n+1} - u_{\mathcal{L}}^{n+1})$$
$$+ \Delta t \sum_{\substack{\sigma \subset \partial \mathcal{K} \\ \sigma \notin \partial \Lambda}} m_\sigma v_{\mathcal{K},\sigma}^{n+1} f(u_\sigma^{n+1}) = m_{\mathcal{K}} \beta(u_{\mathcal{K}}^{n+1}) \Bigg)$$

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$$\sum_{\mathcal{K}\in\mathcal{T}} m_{\mathcal{K}} u_{\mathcal{K}}^{n+1}(u_{\mathcal{K}}^{n+1} - u_{\mathcal{K}}^n) + \Delta t \sum_{\mathcal{K}\in\mathcal{T}} u_{\mathcal{K}}^{n+1} \sum_{\substack{\sigma\subset\partial\mathcal{K}\\ \sigma\notin\partial\Lambda}} \frac{m_\sigma}{d_{\mathcal{K},\mathcal{L}}}(u_{\mathcal{K}}^{n+1} - u_{\mathcal{L}}^{n+1})$$
$$+ \Delta t \sum_{\mathcal{K}\in\mathcal{T}} u_{\mathcal{K}}^{n+1} \sum_{\substack{\sigma\subset\partial\mathcal{K}\\ \sigma\notin\partial\Lambda}} m_\sigma v_{\mathcal{K},\sigma}^{n+1} f(u_\sigma^{n+1}) = \Delta t \sum_{\mathcal{K}\in\mathcal{T}} m_{\mathcal{K}} \beta(u_{\mathcal{K}}^{n+1}) u_{\mathcal{K}}^{n+1}$$

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- Scheme

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$$+ \Delta t \sum_{\mathcal{K}\in\mathcal{T}} u_{\mathcal{K}}^{n+1} \sum_{\substack{\sigma\subset\partial\mathcal{K} \\ \sigma\notin\partial\Lambda}} m_\sigma v_{\mathcal{K},\sigma}^{n+1} f(u_\sigma^{n+1}) = \Delta t \sum_{\mathcal{K}\in\mathcal{T}} m_{\mathcal{K}} \beta(u_{\mathcal{K}}^{n+1}) u_{\mathcal{K}}^{n+1}$$

$$\Rightarrow \frac{1}{2}\left(\left\| u_{\mathcal{T}}^{n+1} \right\|_{L^2(\Lambda)}^2 - \|u_{\mathcal{T}}^n\|_{L^2(\Lambda)}^2 + \left\| u_{\mathcal{T}}^{n+1} - u_{\mathcal{T}}^n \right\|_{L^2(\Lambda)}^2 \right) + \Delta t \left| u_{\mathcal{T}}^{n+1} \right|_{1,\mathcal{T}}^2$$

$$+ \Delta t \sum_{\mathcal{K}\in\mathcal{T}} u_{\mathcal{K}}^{n+1} \sum_{\substack{\sigma\subset\partial\mathcal{K} \\ \sigma\notin\partial\Lambda}} m_\sigma v_{\mathcal{K},\sigma}^{n+1} f(u_\sigma^{n+1}) = \Delta t \sum_{\mathcal{K}\in\mathcal{T}} m_{\mathcal{K}} \beta(u_{\mathcal{K}}^{n+1}) u_{\mathcal{K}}^{n+1}$$

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Let $u_{\mathcal{T}}^n \in \mathbb{R}^{\mathcal{T}}$ be given. We assume there exists a solution $u_{\mathcal{T}}^{n+1}$ to the scheme, then there exists $M > 0$ independent of $\mathrm{size}(\mathcal{T})$ and Δt s.t.:

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Proof:

- Scheme

$$\frac{1}{2}\left(\left\|u_{\mathcal{T}}^{n+1}\right\|_{L^2(\Lambda)}^2 - \|u_{\mathcal{T}}^n\|_{L^2(\Lambda)}^2 + \left\|u_{\mathcal{T}}^{n+1} - u_{\mathcal{T}}^n\right\|_{L^2(\Lambda)}^2\right) + \Delta t \left|u_{\mathcal{T}}^{n+1}\right|_{1,\mathcal{T}}^2$$
$$+ \Delta t \sum_{\mathcal{K} \in \mathcal{T}} u_{\mathcal{K}}^{n+1} \sum_{\substack{\sigma \subset \partial\mathcal{K} \\ \sigma \notin \partial\Lambda}} m_\sigma v_{\mathcal{K},\sigma}^{n+1} f(u_\sigma^{n+1}) = \Delta t \sum_{\mathcal{K} \in \mathcal{T}} m_{\mathcal{K}} \beta(u_{\mathcal{K}}^{n+1}) u_{\mathcal{K}}^{n+1}$$

- Convective term

PROPOSITION (BOUNDS ON THE SOLUTION)

Let $u_\mathcal{T}^n \in \mathbb{R}^\mathcal{T}$ be given. We assume there exists a solution $u_\mathcal{T}^{n+1}$ to the scheme, then there exists $M > 0$ independent of $\mathrm{size}(\mathcal{T})$ and Δt s.t.:

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PROOF:

- Scheme

$$\frac{1}{2}\left(\left\| u_\mathcal{T}^{n+1} \right\|_{L^2(\Lambda)}^2 - \|u_\mathcal{T}^n\|_{L^2(\Lambda)}^2 + \left\| u_\mathcal{T}^{n+1} - u_\mathcal{T}^n \right\|_{L^2(\Lambda)}^2 \right) + \Delta t \left| u_\mathcal{T}^{n+1} \right|_{1,\mathcal{T}}^2$$
$$+ \Delta t \sum_{\mathcal{K} \in \mathcal{T}} u_\mathcal{K}^{n+1} \sum_{\substack{\sigma \subset \partial \mathcal{K} \\ \sigma \notin \partial \Lambda}} m_\sigma v_{\mathcal{K},\sigma}^{n+1} f(u_\sigma^{n+1}) = \Delta t \sum_{\mathcal{K} \in \mathcal{T}} m_\mathcal{K} \beta(u_\mathcal{K}^{n+1}) u_\mathcal{K}^{n+1}$$

- Convective term

$$\sum_{\substack{\sigma \subset \partial \mathcal{K} \\ \sigma \notin \partial \Lambda}} m_\sigma v_{\mathcal{K},\sigma}^{n+1} = \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \sum_{\substack{\sigma \subset \partial \mathcal{K} \\ \sigma \notin \partial \Lambda}} \int_\sigma \mathbf{v}(t,x) \cdot \mathbf{n}_{\boldsymbol{\sigma}\boldsymbol{\mathcal{K}}}(x) \mathrm{d}\gamma(x) \mathrm{d}t = 0.$$

Proposition (Bounds on the solution)

Let $u_{\mathcal{T}}^n \in \mathbb{R}^{\mathcal{T}}$ be given. We assume there exists a solution $u_{\mathcal{T}}^{n+1}$ to the scheme, then there exists $M > 0$ independent of $\mathrm{size}(\mathcal{T})$ and Δt s.t.:

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Proof:

- Scheme

$$\frac{1}{2}\left(\left\|u_{\mathcal{T}}^{n+1}\right\|_{L^2(\Lambda)}^2 - \|u_{\mathcal{T}}^n\|_{L^2(\Lambda)}^2 + \left\|u_{\mathcal{T}}^{n+1} - u_{\mathcal{T}}^n\right\|_{L^2(\Lambda)}^2\right) + \Delta t\left|u_{\mathcal{T}}^{n+1}\right|_{1,\mathcal{T}}^2$$
$$+ \Delta t \sum_{\mathcal{K}\in\mathcal{T}} u_{\mathcal{K}}^{n+1} \sum_{\substack{\sigma\subset\partial\mathcal{K} \\ \sigma\notin\partial\Lambda}} m_\sigma v_{\mathcal{K},\sigma}^{n+1} f(u_\sigma^{n+1}) = \Delta t \sum_{\mathcal{K}\in\mathcal{T}} m_{\mathcal{K}} \beta(u_{\mathcal{K}}^{n+1}) u_{\mathcal{K}}^{n+1}$$

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$$\sum_{\substack{\sigma\subset\partial\mathcal{K} \\ \sigma\notin\partial\Lambda}} m_\sigma v_{\mathcal{K},\sigma}^{n+1} = \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} \underbrace{\sum_{\substack{\sigma\subset\partial\mathcal{K} \\ \sigma\notin\partial\Lambda}} \int_\sigma \mathbf{v}(t,x)\cdot\mathbf{n}_{\sigma\mathcal{K}}(x)\mathrm{d}\gamma(x)}_{=\int_{\partial\mathcal{K}} \mathbf{v}(t,x)\cdot\mathbf{n}_{\sigma\mathcal{K}}(x)\mathrm{d}\gamma(x)=\int_{\mathcal{K}} \mathrm{div}(\mathbf{v})=0} \mathrm{d}t = 0.$$

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Proof:

- Scheme

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- Convective term

$$\sum_{\substack{\sigma \subset \partial\mathcal{K} \\ \sigma \notin \partial\Lambda}} m_\sigma v_{\mathcal{K},\sigma}^{n+1} = \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \underbrace{\sum_{\substack{\sigma \subset \partial\mathcal{K} \\ \sigma \notin \partial\Lambda}} \int_\sigma \mathbf{v}(t,x) \cdot \mathbf{n}_{\boldsymbol{\sigma}\boldsymbol{\mathcal{K}}}(x) \mathrm{d}\gamma(x)}_{= \int_{\partial\mathcal{K}} \mathbf{v}(t,x) \cdot \mathbf{n}_{\boldsymbol{\sigma}\boldsymbol{\mathcal{K}}}(x) \mathrm{d}\gamma(x) = \int_{\mathcal{K}} \mathrm{div}(\mathbf{v}) = 0} \mathrm{d}t = 0.$$

$$\boxed{\sum_{\substack{\sigma \subset \partial\mathcal{K} \\ \sigma \notin \partial\Lambda}} m_\sigma v_{\mathcal{K},\sigma}^{n+1} f(u_\sigma^{n+1}) = \sum_{\substack{\sigma \subset \partial\mathcal{K} \\ \sigma \notin \partial\Lambda}} m_\sigma v_{\mathcal{K},\sigma}^{n+1} \left(f(u_\sigma^{n+1}) - f(u_{\mathcal{K}}^{n+1}) \right).}$$

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Proof:

- Scheme

$$\frac{1}{2}\left(\left\|u_{\mathcal{T}}^{n+1}\right\|_{L^2(\Lambda)}^2 - \|u_{\mathcal{T}}^n\|_{L^2(\Lambda)}^2 + \left\|u_{\mathcal{T}}^{n+1} - u_{\mathcal{T}}^n\right\|_{L^2(\Lambda)}^2\right) + \Delta t \left|u_{\mathcal{T}}^{n+1}\right|_{1,\mathcal{T}}^2$$
$$+ \Delta t \sum_{\mathcal{K}\in\mathcal{T}} u_{\mathcal{K}}^{n+1} \sum_{\substack{\sigma\subset\partial\mathcal{K} \\ \sigma\notin\partial\Lambda}} m_\sigma v_{\mathcal{K},\sigma}^{n+1} f(u_\sigma^{n+1}) = \Delta t \sum_{\mathcal{K}\in\mathcal{T}} m_{\mathcal{K}} \beta(u_{\mathcal{K}}^{n+1}) u_{\mathcal{K}}^{n+1}$$

- Convective term

$$\sum_{\substack{\sigma\subset\partial\mathcal{K} \\ \sigma\notin\partial\Lambda}} m_\sigma v_{\mathcal{K},\sigma}^{n+1} f(u_\sigma^{n+1}) = \sum_{\substack{\sigma\subset\partial\mathcal{K} \\ \sigma\notin\partial\Lambda}} m_\sigma v_{\mathcal{K},\sigma}^{n+1} \left(f(u_\sigma^{n+1}) - f(u_{\mathcal{K}}^{n+1})\right)$$
$$= \sum_{\substack{\sigma\subset\partial\mathcal{K} \\ \sigma\notin\partial\Lambda}} m_\sigma (v_{\mathcal{K},\sigma}^{n+1})^- \left(f(u_{\mathcal{K}}^{n+1}) - f(u_{\mathcal{L}}^{n+1})\right)$$

Proposition (Bounds on the solution)

Let $u_{\mathcal{T}}^n \in \mathbb{R}^{\mathcal{T}}$ be given. We assume there exists a solution $u_{\mathcal{T}}^{n+1}$ to the scheme, then there exists $M > 0$ independent of $\mathrm{size}(\mathcal{T})$ and Δt s.t.:

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Proof:

- Scheme

$$\frac{1}{2}\left(\left\|u_{\mathcal{T}}^{n+1}\right\|_{L^2(\Lambda)}^2 - \|u_{\mathcal{T}}^n\|_{L^2(\Lambda)}^2 + \left\|u_{\mathcal{T}}^{n+1} - u_{\mathcal{T}}^n\right\|_{L^2(\Lambda)}^2\right) + \Delta t\left|u_{\mathcal{T}}^{n+1}\right|_{1,\mathcal{T}}^2 + \Delta t \sum_{\mathcal{K}\in\mathcal{T}} u_{\mathcal{K}}^{n+1} \sum_{\substack{\sigma\subset\partial\mathcal{K}\\ \sigma\notin\partial\Lambda}} m_\sigma v_{\mathcal{K},\sigma}^{n+1} f(u_\sigma^{n+1}) = \Delta t \sum_{\mathcal{K}\in\mathcal{T}} m_{\mathcal{K}} \beta(u_{\mathcal{K}}^{n+1}) u_{\mathcal{K}}^{n+1}$$

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$$\sum_{\mathcal{K}\in\mathcal{T}} u_{\mathcal{K}}^{n+1} \sum_{\substack{\sigma\subset\partial\mathcal{K}\\ \sigma\notin\partial\Lambda}} m_\sigma v_{\mathcal{K},\sigma}^{n+1} f(u_\sigma^{n+1}) = \sum_{\mathcal{K}\in\mathcal{T}} \sum_{\substack{\sigma\subset\partial\mathcal{K}\\ \sigma\notin\partial\Lambda}} m_\sigma (v_{\mathcal{K},\sigma}^{n+1})^- \left(f(u_{\mathcal{K}}^{n+1}) - f(u_{\mathcal{L}}^{n+1})\right) u_{\mathcal{K}}^{n+1}$$

Proposition (Bounds on the solution)

Let $u^n_{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}$ be given. We assume there exists a solution $u^{n+1}_{\mathcal{T}}$ to the scheme, then there exists $M > 0$ independent of $\mathrm{size}(\mathcal{T})$ and Δt s.t.:

$$\sup_{n \le N} \|u^n_{\mathcal{T}}\|_{L^2(\Lambda)} \le M \quad \text{and} \quad \sum_{n=0}^{N-1} \Delta t \left| u^{n+1}_{\mathcal{T}} \right|^2_{1,\mathcal{T}} \le M.$$

Proof:

- Scheme

$$\frac{1}{2}\left(\left\| u^{n+1}_{\mathcal{T}} \right\|^2_{L^2(\Lambda)} - \|u^n_{\mathcal{T}}\|^2_{L^2(\Lambda)} + \left\| u^{n+1}_{\mathcal{T}} - u^n_{\mathcal{T}} \right\|^2_{L^2(\Lambda)} \right) + \Delta t \left| u^{n+1}_{\mathcal{T}} \right|^2_{1,\mathcal{T}}$$
$$+ \Delta t \sum_{\mathcal{K} \in \mathcal{T}} u^{n+1}_{\mathcal{K}} \sum_{\substack{\sigma \subset \partial\mathcal{K} \\ \sigma \notin \partial\Lambda}} m_\sigma v^{n+1}_{\mathcal{K},\sigma} f(u^{n+1}_\sigma) = \Delta t \sum_{\mathcal{K} \in \mathcal{T}} m_{\mathcal{K}} \beta(u^{n+1}_{\mathcal{K}}) u^{n+1}_{\mathcal{K}}$$

- Convective term $\qquad \forall r \in \mathbb{R},\ \Phi(r) = \int_0^r f'(s) s \, ds$

$$\sum_{\mathcal{K} \in \mathcal{T}} u^{n+1}_{\mathcal{K}} \sum_{\substack{\sigma \subset \partial\mathcal{K} \\ \sigma \notin \partial\Lambda}} m_\sigma v^{n+1}_{\mathcal{K},\sigma} f(u^{n+1}_\sigma) = \sum_{\mathcal{K} \in \mathcal{T}} \sum_{\substack{\sigma \subset \partial\mathcal{K} \\ \sigma \notin \partial\Lambda}} m_\sigma (v^{n+1}_{\mathcal{K},\sigma})^- \left(f(u^{n+1}_{\mathcal{K}}) - f(u^{n+1}_{\mathcal{L}}) \right) u^{n+1}_{\mathcal{K}}$$

$$b(f(b) - f(a)) = \int_a^b (s f(s))' ds - (b-a) f(a) = \int_a^b \Phi'(s) ds + \int_a^b (f(s) - f(a)) ds$$
$$\ge (\Phi(b) - \Phi(a)) + \frac{1}{2L_f}(f(b) - f(a))^2 \ge (\Phi(b) - \Phi(a)).$$

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PROOF:

- Scheme

$$\frac{1}{2}\left(\left\|u_{\mathcal{T}}^{n+1}\right\|_{L^2(\Lambda)}^2 - \|u_{\mathcal{T}}^n\|_{L^2(\Lambda)}^2 + \left\|u_{\mathcal{T}}^{n+1} - u_{\mathcal{T}}^n\right\|_{L^2(\Lambda)}^2\right) + \Delta t\left|u_{\mathcal{T}}^{n+1}\right|_{1,\mathcal{T}}^2 + \Delta t \sum_{\mathcal{K}\in\mathcal{T}} u_{\mathcal{K}}^{n+1} \sum_{\substack{\sigma\subset\partial\mathcal{K}\\ \sigma\notin\partial\Lambda}} m_\sigma v_{\mathcal{K},\sigma}^{n+1} f(u_\sigma^{n+1}) = \Delta t \sum_{\mathcal{K}\in\mathcal{T}} m_{\mathcal{K}} \beta(u_{\mathcal{K}}^{n+1}) u_{\mathcal{K}}^{n+1}$$

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$$\begin{aligned}\sum_{\mathcal{K}\in\mathcal{T}} u_{\mathcal{K}}^{n+1} \sum_{\substack{\sigma\subset\partial\mathcal{K}\\ \sigma\notin\partial\Lambda}} m_\sigma v_{\mathcal{K},\sigma}^{n+1} f(u_\sigma^{n+1}) &= \sum_{\mathcal{K}\in\mathcal{T}} \sum_{\substack{\sigma\subset\partial\mathcal{K}\\ \sigma\notin\partial\Lambda}} m_\sigma (v_{\mathcal{K},\sigma}^{n+1})^- \left(f(u_{\mathcal{K}}^{n+1}) - f(u_{\mathcal{L}}^{n+1})\right) u_{\mathcal{K}}^{n+1}\\ &\geq \sum_{\mathcal{K}\in\mathcal{T}} \sum_{\substack{\sigma\subset\partial\mathcal{K}\\ \sigma\notin\partial\Lambda}} m_\sigma (v_{\mathcal{K},\sigma}^{n+1})^- \left(\Phi(u_{\mathcal{K}}^{n+1}) - \Phi(u_{\mathcal{L}}^{n+1})\right)\\ &= -\sum_{\mathcal{K}\in\mathcal{T}} \Phi(u_{\mathcal{K}}^{n+1}) \sum_{\substack{\sigma\subset\partial\mathcal{K}\\ \sigma\notin\partial\Lambda}} m_\sigma v_{\mathcal{K},\sigma}^{n+1} = 0.\end{aligned}$$

Proposition (Bounds on the solution)

Let $u_{\mathcal{T}}^n \in \mathbb{R}^{\mathcal{T}}$ be given. We assume there exists a solution $u_{\mathcal{T}}^{n+1}$ to the scheme, then there exists $M > 0$ independent of $\mathrm{size}(\mathcal{T})$ and Δt s.t.:

$$\sup_{n \leq N} \|u_{\mathcal{T}}^n\|_{L^2(\Lambda)} \leq M \quad \text{and} \quad \sum_{n=0}^{N-1} \Delta t \left| u_{\mathcal{T}}^{n+1} \right|_{1,\mathcal{T}}^2 \leq M.$$

Proof:

- Scheme

$$\frac{1}{2}\left(\left\| u_{\mathcal{T}}^{n+1} \right\|_{L^2(\Lambda)}^2 - \|u_{\mathcal{T}}^n\|_{L^2(\Lambda)}^2 + \left\| u_{\mathcal{T}}^{n+1} - u_{\mathcal{T}}^n \right\|_{L^2(\Lambda)}^2 \right) + \Delta t \left| u_{\mathcal{T}}^{n+1} \right|_{1,\mathcal{T}}^2 \leq \Delta t \sum_{\mathcal{K}\in\mathcal{T}} m_{\mathcal{K}} \beta(u_{\mathcal{K}}^{n+1}) u_{\mathcal{K}}^{n+1}$$

- Right-hand side

$$\sum_{\mathcal{K}\in\mathcal{T}} m_{\mathcal{K}} \beta(u_{\mathcal{K}}^{n+1}) u_{\mathcal{K}}^{n+1} \leq L_\beta \left\| u_{\mathcal{T}}^{n+1} \right\|_{L^2(\Lambda)}^2 .$$

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$$\Rightarrow (1-2L_\beta \Delta t)\left(\left\|u_\mathcal{T}^{n+1}\right\|_{L^2(\Lambda)}^2 - \|u_\mathcal{T}^n\|_{L^2(\Lambda)}^2\right) + 2\Delta t \left|u_\mathcal{T}^{n+1}\right|_{1,\mathcal{T}}^2 \leq 2L_\beta \Delta t \|u_\mathcal{T}^n\|_{L^2(\Lambda)}^2$$

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- Conclusion

$$(1-2L_\beta\Delta t)\left\|u_\mathcal{T}^N\right\|_{L^2(\Lambda)}^2 + 2\sum_{n=0}^{N-1} \Delta t \left|u_\mathcal{T}^{n+1}\right|_{1,\mathcal{T}}^2 \leq 2L_\beta \sum_{n=0}^{N-1} \Delta t \|u_\mathcal{T}^n\|_{L^2(\Lambda)}^2 + (1-2L_\beta\Delta t)\left\|u_\mathcal{T}^0\right\|_{L^2(\Lambda)}^2$$

$\rightsquigarrow$ Δt small enough + Gronwall lemma

Existence of a discrete solution

- Topological degree theory:

$$\mathcal{P} : \text{Find } u \in W \text{ s.t. } \mathcal{F}(u) = 0.$$

For any $\delta \in [0,1]$, let $\mathcal{P}_\delta$ be s.t.,

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1. $\mathcal{P}_1 = \mathcal{P}$;
2. $\delta = 0$: Linear system in finite dimension

$$m_\mathcal{K} w_\mathcal{K} + \Delta t \sum_{\substack{\sigma\subset\partial\mathcal{K} \\ \sigma\notin\partial\Lambda}} \frac{m_\sigma}{d_{\mathcal{K},\mathcal{L}}}(w_\mathcal{K} - w_\mathcal{L}) = m_\mathcal{K} u_\mathcal{K}^n.$$

$\Rightarrow \exists!$ a solution iif 0 is the unique solution of the homogeneous system.

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We multiply the scheme by $w_\mathcal{K}$ and sum over $\mathcal{K} \in \mathcal{T}$:

$$\|w_\mathcal{T}\|^2_{L^2(\Lambda)} + \Delta t\, |w_\mathcal{T}|^2_{1,\mathcal{T}} = 0 \;\Rightarrow\; w_\mathcal{T} = 0.$$

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1. $\mathcal{P}_1 = \mathcal{P}$;
2. $\delta = 0$: $\exists!$ solution $w_\mathcal{T} \in \mathbb{R}^\mathcal{T}$ to the discrete problem $\mathcal{P}_0$;
3. $\delta \in [0,1]$: Bounds on the solution

$$(1 - 2L_\beta \Delta t)\, \|w_\mathcal{T}\|_{L^2(\Lambda)}^2 + 2\Delta t\, |w_\mathcal{T}|_{1,\mathcal{T}}^2 \leq \|u_\mathcal{T}^n\|_{L^2(\Lambda)}^2$$

Δt small enough: $(1 - 2L_\beta \Delta t) \geq 1/2$

$$\Rightarrow \|w_\mathcal{T}\|_W^2 = \|w_\mathcal{T}\|_{L^2(\Lambda)}^2 + |w_\mathcal{T}|_{1,\mathcal{T}}^2 \leq 2\, \|u_\mathcal{T}^n\|_{L^2(\Lambda)}^2 + \frac{1}{2\Delta t}\, \|u_\mathcal{T}^n\|_{L^2(\Lambda)}^2 .$$

Weak convergence of the gradient

(Eymard-Gallouët, '02)

Theorem (Weak compactness)

Let $(\mathcal{T}_m)_m$ *be a sequence of admissible orthogonal meshes such that* $\text{size}(\mathcal{T}_m) \to 0$ *and* $(u^{\mathcal{T}_m})_m$ *a family of discrete functions defined on each of these meshes and such that there exists* $M > 0$ *satisfying*

$$\|u_{\mathcal{T}_m}\|_{L^2(\Lambda)} + |u_{\mathcal{T}_m}|_{1,\mathcal{T}} \leq M, \quad \forall m \geq 0.$$

Then

- *There exists a function* $u \in H^1(\Omega)$ *and a subsequence* $(u^{\mathcal{T}_{\phi(m)}})_m$ *that* **weakly** *converges towards* u *in* $L^q(\Omega)$ $(q \geq 1)$.
- *The sequence of discrete gradients* $(\nabla^{\mathcal{T}}_{\phi(m)} u^{\mathcal{T}_{\phi(m)}})_m$ **weakly** *converges towards* ∇u *in* $(L^2(\Omega))^d$.

WEAK CONVERGENCE OF THE GRADIENT **(Eymard-Gallouët, '02)**

THEOREM (WEAK COMPACTNESS)

Let $(\mathcal{T}_m)_m$ be a sequence of admissible orthogonal meshes such that $\text{size}(\mathcal{T}_m) \to 0$ *and $(u^{\mathcal{T}_m})_m$ a family of discrete functions defined on each of these meshes and such that there exists $M > 0$ satisfying*

$$\|u_{\mathcal{T}_m}\|_{L^2(\Lambda)} + |u_{\mathcal{T}_m}|_{1,\mathcal{T}} \leq M, \quad \forall m \geq 0.$$

Then

- *There exists a function $u \in H^1(\Omega)$ and a subsequence $(u^{\mathcal{T}_{\phi(m)}})_m$ that* **weakly** *converges towards u in $L^q(\Omega)$ $(q \geq 1)$.*
- *The sequence of discrete gradients $(\nabla^{\mathcal{T}}_{\phi(m)} u^{\mathcal{T}_{\phi(m)}})_m$* **weakly** *converges towards ∇u in $(L^2(\Omega))^d$.*

Bounds on the solutions $\Rightarrow \exists u \in L^2(0,T;H^1(\Omega))$ such that (up to a subsequence),

$$u^{\Delta t}_{\mathcal{T}} \xrightarrow[\text{size}(\mathcal{T}),\Delta t \to 0]{} u \text{ weakly in } L^2(0,T;L^q(\Omega)),$$

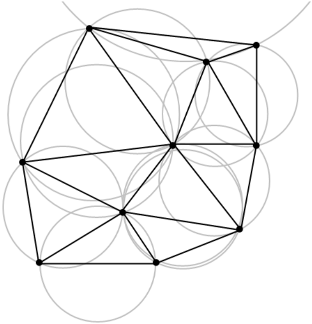
$$\nabla^{\mathcal{T}} u^{\Delta t}_{\mathcal{T}} \xrightarrow[\text{size}(\mathcal{T}),\Delta t \to 0]{} \nabla u \text{ weakly in } L^2(0,T;\Omega).$$

+ Compactness argument for nonlinear term

- Cartesian meshes: Control volumes are rectangular parallelepipeds thus choosing $x_{\mathcal{K}}$ as the mass center is OK

- Cartesian meshes: OK
- Conforming triangular meshes:
  We take $x_\mathcal{K}$ =circumcenter; **BUT :**
  - It is not guaranteed that $x_\mathcal{K} \in \mathcal{K}$ (even $x_\mathcal{K} \in \Omega$ is not sure).
  - We can have $x_\mathcal{K} = x_\mathcal{L}$ for $\mathcal{K} \neq \mathcal{L} \Rightarrow d_{\mathcal{K},\mathcal{L}} = 0$!
  - However, the TPFA scheme still works for the Laplace equation if

$$(x_\mathcal{L} - x_\mathcal{K}) \cdot \vec{\mathbf{n}}_{\mathcal{K}\mathcal{L}} > 0 \quad \Leftrightarrow \text{Delaunay condition}$$



- Cartesian meshes: OK
- Conforming triangular meshes: OK with Delaunay conditions
- For a non conforming triangle mesh: orthogonality condition is impossible to fulfill.
- For a non Cartesian quadrangle mesh: orthogonality condition is impossible to fulfill in general.

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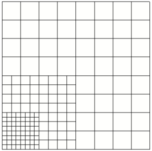
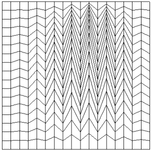
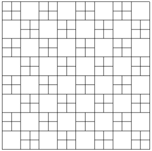
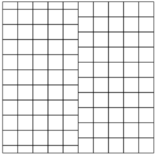
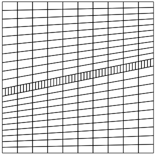
Some academic meshes that we would like to deal with









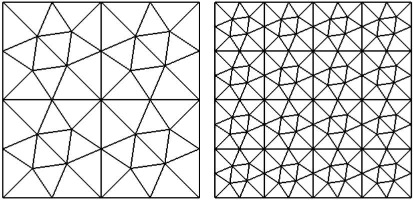


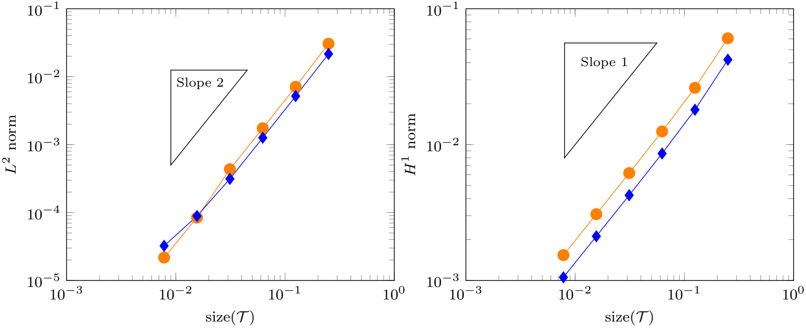
The Laplace problem - Exact solution:

$u(x, y) = \sin(\pi x)\sin(\pi y)$

Centers:

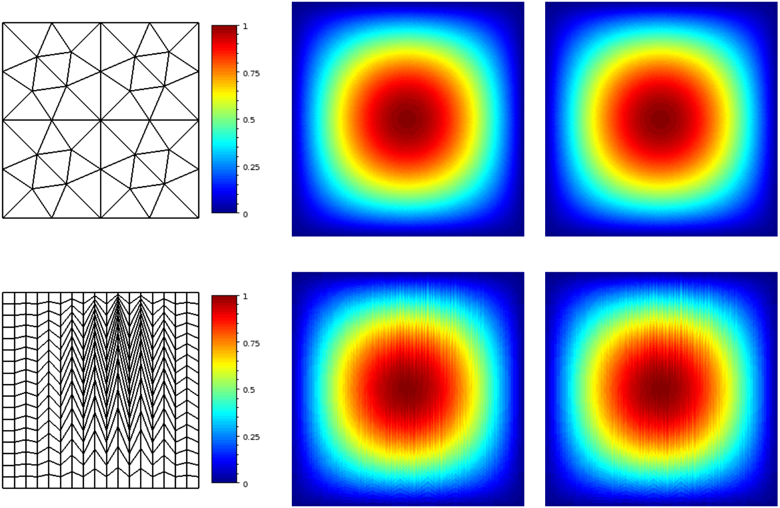
- Circumcenter
- Center of mass





Error L^2 / H^1

The Laplace problem with the DDFV scheme



Mesh Exact solution Approximate solution