On the asymptotic behaviors of solutions to stochastic obstacle problems

Large deviations & invariant measures

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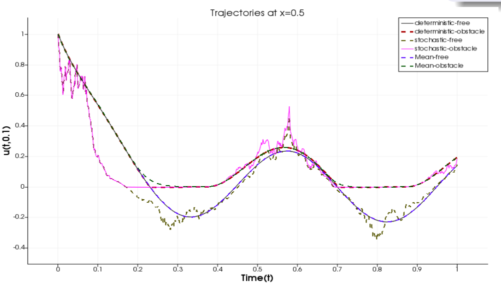
Obstacle problems

$\partial I_K(u) \ni$
$f - \partial_t\left(u - \int_0^{\cdot} G(u,\cdot)dW\right) - A(u,\cdot),$
K : convex in some B-space

$0 = \partial_t\left(u - \int_0^{\cdot} G(u,\cdot)dW\right) + A(u,\cdot) - f$
on $\{u > \psi\}$,
$k = \partial_t\left(u - \int_0^{\cdot} G(u,\cdot)dW\right) + A(u,\cdot) - f$
on $\{u = \psi\}$.



$\overset{(i)}{\rightsquigarrow}$ Formulations? Exist.+ Uniq.?
$\overset{(ii)}{\rightsquigarrow}$ Lewy-Stampacchia inequalities?

$$0 \le -k \le h^-$$

$\overset{(iii)}{\rightsquigarrow}$ More on the dynamics??

- Rare events/ small noise LDP:
  $\rightsquigarrow$ How the stochastic perturbations effects the dynamics? $(u_\delta)_{\delta\downarrow 0?}$

$$\partial I_K(u_\delta) \ni f - \partial_t\Big(u_\delta - \delta\int_0^\cdot G(u_\delta,\cdot)dW\Big) - A(u_\delta,\cdot); \quad \delta > 0, t \geq 0,$$

- Invariant measures & Ergodicity?
  $\rightsquigarrow$ Markov semigroup properties? $t \to +\infty$?

$$\partial I_K(u) \ni f - \partial_t\Big(u - \int_0^\cdot G(u)dW\Big) - A(u), \tag{1}$$

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Singularities caused by the obstacle? $\leftrightsquigarrow$ Lewy-Stampacchia inequality ✓

Large deviations

$(X_t^\delta)' = b(X_t^\delta) + \delta\psi_t, \quad X_t^\delta(0) = x_0 \leftrightsquigarrow (x_t)' = b(x_t), \quad x_t(0) = x_0$

Let $D \subset \mathbb{R}^d$ where $x_0 \in D$.

$x_t \in D$ for any $t \geq 0$ and attracted to equilibrium position x_* as $t \to \infty$.

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$\rightsquigarrow$ Does X_t^δ have the same property with $P \approx 1$?

$\rightsquigarrow P(\sup_{t\in[0,\infty]} |\psi_t| < \infty) = 0 \to I = [0, \infty[= \cup_k I_T^k = \cup_k [kT, (k+1)T]$

$\rightsquigarrow \sup_{t \in I_T^k} |X_t^\delta - x_t| \leq \delta, \quad \delta << 1$

$\to P(X_t^\delta \notin D \text{ for some } t \in [kT, (k+1)T]) \approx \delta$

$\rightarrow$ The behaviour of X_t^δ on I_T^k are independent $\Longrightarrow$ $X_t^\delta \notin D$.

$\rightarrow$ The first time exit $\tau^\delta \sim Exp(\lambda)$ with

$$\lambda = \frac{1}{T} P(X_t^\delta \text{ exists from } D \text{ for } t \in I_T^k)$$

X_t^δ exists D with $P \approx \delta \leftrightsquigarrow$ the probability of improbable events $t \nearrow +\infty$?

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- Gaussian perturbations $\leftrightsquigarrow$ asymptotics of the form $exp(\frac{-c}{\delta^2}), \delta \downarrow 0$

$$P(d(X_t^\delta, \varphi) < \delta) \approx exp(\frac{-1}{\delta^2} S(\varphi)) \quad for \quad \delta, \delta << 1.$$

"φ is a smooth function"

S : rate function $\leftrightsquigarrow$ Entropy in statistical mechanics.

Definitions

Let E be a Polish space with the Borel $\sigma-$field $\mathcal{B}(E)$.

Definition

A function $I : E \to [0, \infty]$ is called a rate function on E if for each $M < \infty$ the level set $\{x \in E : I(x) \leq M\}$ is a compact.

Definition

A family $\{X^\delta\}_\delta$ of $E-$valued random elements is said to satisfy a large deviation principle with rate function I if for each $G \in \mathcal{B}(E)$

$$
\begin{aligned}
-\inf_{x \in G^0} I(x) \leq \liminf_{\delta \to 0} \delta^2 \log P(X^\delta \in G) &\leq \limsup_{\delta \to 0} \delta^2 \log P(X^\delta \in G) \\
&\leq -\inf_{x \in \bar{G}} I(x),
\end{aligned}
$$

where G^0 and $\bar{G}$ are respectively the interior and the closure of G in E.

LDP / infinite dimensional setting

- The variational representation for functionals of Wiener process and LDP $\rightsquigarrow$ A.Budhiraja and P.Dupuis 20':

  LDP and the Laplace principle, weak cv method.

- Sufficient condition to prove LDP: Matoussi, Sabbagh and Zhang 21' $\rightsquigarrow$ Reflected measure, penalization, weak cv.

  $\rightsquigarrow$ Quasilinear obstacle problems!

Notations

For $N \in \mathbb{N}$:

$$S_N = \Big\{\phi \in L^2([0,T];H_0); \quad \int_0^T \|\phi(s)\|^2_{H_0} ds \le N\Big\},$$

$$\mathcal{A} = \Big\{v; \quad v : \Omega_T \to H_0, \text{ predictable } P\big(\int_0^T \|v(s)\|^2_{H_0} ds < \infty\big) = 1\Big\},$$

$$\mathcal{A}_N = \{v \in \mathcal{A} : v(\omega) \in S_N \text{ P-a.s.}\}.$$

$\rightsquigarrow g^\delta$, g^0?

Let $g^\delta : C([0,T],H) \to E$ be a measurable map such that $g^\delta(W(\cdot)) = X^\delta$. Suppose that there exists a measurable map $g^0 : C([0,T],H) \to E$ s.t.

1. For every $N < \infty$, any family $\{v^\delta : \delta > 0\} \subset \mathcal{A}_N$ and any $\epsilon > 0$,

   $$\lim_{\delta \to 0} P(\rho(Y^\delta, Z^\delta) > \epsilon) = 0,$$

   where $Y^\delta = g^\delta(W(\cdot) + \frac{1}{\delta}\int_0^\cdot v_s^\delta ds)$, $Z^\delta = g^0(\int_0^\cdot v_s^\delta ds)$ and $\rho(\cdot,\cdot)$ stands for the metric in the space E.

2. For every $N < \infty$ and any family $\{v^\delta : \delta > 0\} \subset S_N$ satisfying that v_δ converges weakly to some element v as $\delta \to 0$,

   $$g^0(\int_0^\cdot v_s^\delta ds) \to g^0(\int_0^\cdot v_s ds) \text{ in the space } E.$$

LDP for obstacle problems $(\mathcal{P})$

$D \subset \mathbb{R}^d$, $T > 0$; $V = W_0^{1,p}(D)$; $H = L^2(D)$, $(\Omega, \mathcal{F}, P; \{\mathcal{F}_t\}_{t\geq 0})$ is given.

$$\begin{cases} du_\delta + A(u_\delta, \cdot)ds + k_\delta ds = fds + \delta G(\cdot, u_\delta)dW, \delta > 0 & \text{in} \quad D \times \Omega_T, \\ u_\delta(t=0) = u_0 & \text{in} \quad D, \\ u_\delta \geq \psi & \text{in} \quad D \times \Omega_T, \\ u_\delta = 0 & \text{on} \quad \partial D \times \Omega_T, \\ \langle k_\delta, u_\delta - \psi \rangle_{V',V} = 0 \ \text{ and } \ -k_\delta \in (V')^+ & \text{a.e. in} \quad \Omega_T. \end{cases}$$

$\exists ! X_\delta := (u_\delta, k_\delta)$. In particular, $\{X_\delta\}_\delta$ takes values in

$$(C([0,T];H) \cap L^p(0,T;V)) \times L^{p'}(0,T;V') \quad P - a.s.$$

$(C([0,T];H) \cap L^p(0,T;V), |\cdot|_T)$ is a Polish space with the following norm

$$|f - g|_T = \sup_{t\in[0,T]} \|f - g\|_H + \Big(\int_0^T \|f - g\|_V^p ds\Big)^{1/p}.$$

⇝ Y.Tahraoui and G. Vallet: Lewy–Stampacchia's inequality for a stochastic T-monotone obstacle problem, Stoch PDE: Anal Comp 10, 90–125 (2022).

Assumptions (LDP)

- $W(\cdot)$ is a $\{\mathcal{F}_t\}_{t\geq 0}$-adapted Q-Wiener process in H.
- $\max(1, \frac{2d}{d+2}) < p < +\infty$.

H_1 : Let $A : V \times [0,T] \to V'$, $G : H \times [0,T] \to L_2(\overbrace{H_0}^{Q^{1/2}H}, H)$.

H_2 : A is T-monotone, coercive, hemicontinuous $+$growth.

H_3 : G is Lipschitz w.r.t. $u \in H$ and $\|G(0,\cdot)\|^2_{L_2(H_0,H)} < \infty$.

H_4 : $\psi \in L^\infty(0,T;V)$, $\frac{d\psi}{dt} \in L^{p'}(0,T;V')$ and $G(\psi,\cdot) = 0$.

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$\rightsquigarrow \max(1, \frac{2d}{d+2}) < p < 2!$

H$_5$: $f \in L^\infty(0, T; V')$ s.t. $f - \partial_t\psi - A(\psi, \cdot) \in L^p(0, T; V)^*$.

H$_6$: $u_0 \in H$ satisfies the constraint, *i.e.* $u_0 \geq \psi(0)$.

Main result

✓ the existence of a Borel-measurable map [1]

$$g^\delta : C([0,T];H) \to C([0,T];H) \cap L^p(0,T;V); u_\delta = g^\delta(W) \quad P-a.s.$$

[1] M. Röckner, B. Schmuland and X. Zhang: Yamada-Watanabe theorem for stochastic evolution equations in infinite dimensions. Condens. Matter Phys. 54, 247–259, 2008.

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$\to g^0?$: Let $\phi \in L^2([0,T];H_0)$, $\exists!(y^\phi, R^\phi)$ s.t.

$$\begin{cases} \dfrac{dy^\phi}{dt} + A(y^\phi,\cdot) + R^\phi = f + G(\cdot, y^\phi)\phi \quad \text{in } V', y^\phi(0) = u_0 \text{ in } H, \\ y^\phi \geq \psi \text{ a.e. in } \quad D \times [0,T], \\ -R^\phi \in (V')^+ : \langle R^\phi, y^\phi - \psi \rangle_{V',V} = 0 \text{ a.e. in } [0,T]. \end{cases} \tag{2}$$

"the skeleton equations"

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Definition

The pair (y^ϕ, R^ϕ) is a solution to (2) if and only if:

- $(y^\phi, R^\phi) \in (L^p(0,T;V) \cap C([0,T];H)) \times L^{p'}(0,T;V')$, $y^\phi(0) = u_0$ and $y^\phi \geq \psi$.
- $-R^\phi \in (L^{p'}(0,T;V'))^+$ and $\displaystyle\int_0^T \langle R^\phi, y^\phi - \psi\rangle ds = 0$.
- For all $t \in [0,T]$,

$$(y^\phi(t), \Phi) + \int_0^t \langle R^\phi, \Phi\rangle ds + \int_0^t \langle A(y^\phi, \cdot), \Phi\rangle ds$$
$$= (u_0, \Phi) + \int_0^t (f + G(\cdot, y^\phi)\phi, \Phi) ds, \quad \forall \Phi \in V.$$

Theorem

$\exists!(y^\phi, R^\phi)$ *to* (2). *Moreover, the following L-S inequality holds:*

$$0 \leq \frac{dy^\phi}{ds} + A(y^\phi, \cdot) - G(y^\phi, \cdot)\phi - f \leq h^- = (f - \partial_t\psi - A(\psi, \cdot))^- . \quad (3)$$

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✓ Define the measurable mapping g^0 as follows

$$g^0 : C([0,T];H) \to L^p(0,T;V) \cap C([0,T];H)$$
$$\int_0^\cdot \phi ds \mapsto g^0(\int_0^\cdot \phi ds) := y^\phi \quad \text{for } \phi \in L^2([0,T];H_0), \quad (4)$$

where $y^\phi \in L^p(0,T;V) \cap C([0,T];H)$ is the unique solution of (2).

Result "LDP"

Theorem

Assume H_1-H_6 hold. Let $\{u^\delta\}_{\delta>0}$ be the unique solution of $(\mathcal{P})$. Then

1. $\{u^\delta\}_\delta$ *satisfies LDP on* $C([0,T];H)$, *as* $\delta\downarrow 0$, *with the rate function* $\mathbb{I}$.
2. *If moreover H_7 holds. Then* $\{u^\delta\}_\delta$ *satisfies LDP on* $C([0,T];H)\cap L^p(0,T;V)$, *as* $\delta\downarrow 0$, *with the rate function* $\mathbb{I}$.

$$\mathbb{I}(y)=\inf_{\{\phi\in L^2([0,T],H_0)\}}\{\frac{1}{2}\int_0^T\|\phi(s)\|_{H_0}^2ds;\quad y:=y^\phi=g^0(\int_0^\cdot\phi ds)\}.$$

$$H_7:\exists\lambda_T\in\mathbb{R},\exists\bar{\alpha}>0:\forall v_1,v_2\in V$$

$$\langle A(v_1,\cdot)-A(v_2,\cdot),v_1-v_2\rangle\geq\bar{\alpha}\|v_1-v_2\|_V^p-\lambda_T\|v_1-v_2\|_H^2.$$

Outline of the proof

a- The Skeleton equation

$$\begin{cases} \dfrac{dy^\phi}{dt} + A(y^\phi, \cdot) + R^\phi = f + G(\cdot, y^\phi)\phi \quad \text{ in } V', y^\phi(0) = u_0 \text{ in } H, \\ y^\phi \geq \psi \text{ a.e. in } \quad D \times [0, T], \\ -R^\phi \in (V')^+ : \langle R^\phi, y^\phi - \psi \rangle_{V',V} = 0 \text{ a.e. in } [0, T]. \end{cases}$$

- Uniqueness $\leftrightsquigarrow$ monotonicity+ Lipschitz multiplicative noise.
- Existence & L-S inequalities: Regular data $\rightarrow$ general data

  ✓ Talk by Guy Vallet at 14:00

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- Uniqueness $\leftrightsquigarrow$ monotonicity+ Lipschitz multiplicative noise.
- Existence & L-S inequalities: Regular data $\to$ general data
  ✓ Talk by Guy Vallet at 14:00
- $\max(1, \frac{2d}{d+2}) < p < 2 \leftrightsquigarrow G(\cdot, y^\phi)\phi \leftrightsquigarrow L^\infty(..;)$ in $H_4 + H_5$!

b- Continuity of skeleton equations

Let $\{v^\delta : \delta > 0\} \subset S_N$ satisfying that $v_\delta \rightharpoonup v$ as $\delta \to 0$. Then

- $g^0(\int_0^\cdot v_s^\delta ds)$ converges to $g^0(\int_0^\cdot v_s ds)$ in the space $C([0,T];H)$,
- If moreover H_7 holds, then

$$g^0\Big(\int_0^\cdot v_s^\delta ds\Big) \to g^0\Big(\int_0^\cdot v_s ds\Big) \text{ in } L^p(0,T;V) \cap C([0,T];H).$$

Let $(\phi_n)_n \subset S_N$ such that $\phi_n \rightharpoonup \phi$ in $L^2(0,T;H_0)$.

$$y_n \to y^\phi \text{ in } L^p(0,T;V) \cap C([0,T];H)?$$

y_n and y^ϕ resp. are the solutions of (2) corresponding to ϕ_n and ϕ resp.

- A priori estimates $\leftrightsquigarrow$ L-S inequalities ✓
- Convergence of $y_{n_k} \to y$ in $L^2(0, T; H)$ $\leftrightsquigarrow$ Compactness ✓
- Boundedness of (ϕ_n) & weak cv+ strong cv $y_n \to y$ in $L^2(0, T; H)$.
- Monotonicity $\implies$ convergence of y_{n_k} to $\widetilde{y}$.
- cv of y_n and $\widetilde{y} = y^{\phi}$ $\leftrightsquigarrow$ Uniqueness ✓

c- vanishing multiplicative noise $\delta \downarrow 0$

Assume $H_1 - H_6$. Let $\{\phi^\delta : \delta > 0\} \subset \mathcal{A}_N$ for some $N < \infty$. Then

$$\lim_{\delta \to 0} P\big(\sup_{t \in [0,T]} \|(v_\delta - u_\delta)(t)\|_H > \epsilon \big) = 0, \quad \forall \epsilon > 0.$$

Moreover, if H_7 holds, then $\lim_{\delta \to 0} P(|v_\delta - u_\delta|_T > \epsilon) = 0, \quad \forall \epsilon > 0.$

$$v_\delta = g^\delta\Big(W(\cdot) + \frac{1}{\delta}\int_0^\cdot \phi^\delta(s)ds\Big) = g^\delta(W^\delta(\cdot)),$$
$$u_\delta = g^0(\phi^\delta).$$

Let $\{\phi^\delta\}_{\delta>0} \subset \mathcal{A}_N$ for some $N < \infty$.

- Girsanov theorem $\implies \exists$ Q-Wiener process W^δ, with respect to $\{\mathcal{F}_t\}_{t\geq 0}$ on $(\Omega, \mathcal{F}, P^\delta)$ where
  $W^\delta(t) = W(t) + \frac{1}{\delta}\int_0^t \phi^\delta(s)ds, \quad t \in [0, T]$ and

$$dP^\delta = \exp[-\frac{1}{\delta}\int_0^T \langle \phi^\delta(s), dW(s)\rangle_{H_0} - \frac{1}{2\delta^2}\int_0^T \|\phi^\delta(s)\|_{H_0}^2 ds]dP. \quad (5)$$

$$\rightsquigarrow v_\delta = g^\delta(W(\cdot) + \frac{1}{\delta}\textstyle\int_0^\cdot \phi^\delta(s)ds) = g^\delta(W^\delta(\cdot)).$$

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$$\rightsquigarrow v_\delta = g^\delta(W(\cdot) + \frac{1}{\delta}\textstyle\int_0^\cdot \phi^\delta(s)ds) = g^\delta(W^\delta(\cdot)).$$

- Uniform estimates for (v_δ) and (u_δ): ψ is "deterministic"!

  L-S not preserved $\leftrightsquigarrow$ P^δ!

- Conclusion.

Invariant measures

- Morkovian process
- $\lim_{t\to\infty} P_t^* \delta_{u_0}$ for given transition probability (semigroup) P_t?
- Convergence to equilibrium? (strongly mixing, Exponential$\cdots$).
- Ergodicity: for any $\varphi \in L^2(H, \mu)$

"Temporal" average of $P_t\varphi$ coincides with the "spatial" average of φ.

Invariant measures ($T \to +\infty$)

$$\begin{cases} du + A(u)ds + kds = fds + G(u)dW & \text{in} \quad D \times \Omega_T, \\ u(t=0) = u_0 \text{ in } \quad D; \quad u \geq \psi & \text{in} \quad D \times \Omega_T, \\ u = 0 & \text{on} \quad \partial D \times \Omega_T, \\ \langle k, u - \psi \rangle_{V',V} = 0 \text{ and } -k \in (V')^+ & \text{a.e. in} \quad \Omega_T, \end{cases} \tag{6}$$

- $A : V \to V'$ and $G : H \to L_2(H_0, H)$ are measurables.
- A is T-monotone, coercive, hemicontinuous + growth.
- $\forall \theta, \sigma \in H$: $\|G(\theta) - G(\sigma)\|^2_{L_2(H_0,H)} \leq L_G \|\theta - \sigma\|^2_H$; $\|G(0, \cdot)\|^2_{L_2(H_0,H)} < \infty$.
- $\psi \in V$ and $G(\psi) = 0$, $f \in V'$ such that $f - A(\psi) = h \in V^*$.
- $u_0 \in H$ satisfies the constraint, *i.e.* $u_0 \geq \psi$.

Theorem

$\exists!(u,k)$ *to* (6). *Moreover, the following L-S inequality holds*

$$0 \leq \partial_t\Big(u - \int_0^\cdot G(u)dW\Big) + A(u) - f = -k \leq h^- = (f - A(\psi))^- . \quad (7)$$

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$\rightsquigarrow$ Let $\epsilon > 0$ and consider the following approximating problem:

$$\begin{cases} u_\epsilon(t) + \int_0^t (A(u_\epsilon) - \frac{1}{\epsilon}[(u_\epsilon - \psi)^-]^{\tilde{q}-1} - f)ds = u_0 + \int_0^t \widetilde{G}(u_\epsilon)dW(s) \\ u_0 \ge \psi, \end{cases} \quad (8)$$

where $\tilde{q} = \min(p,2)$ and $\widetilde{G}(u_\epsilon) = G(\max(u_\epsilon,\psi))$.

$\rightsquigarrow$ Let u_ϵ and u, respectively are the unique solution of (8) and (6):

$$E \sup_{t\in[0,T]} \|(u_\epsilon - u)(t)\|_H^2 \le C\epsilon \quad \text{for all } \epsilon > 0, \quad (9)$$

Markov semigroup

Let $\epsilon > 0$ and $u_\epsilon(t, s; \xi), t \geq s \geq 0$ be the unique strong solution to (8) starting at time s from an $\mathcal{F}_s$-measurable initial data $\xi \in H$. Let $T > 0$,

$$\mathbb{E}\|u_\epsilon(t, s; \xi) - u_\epsilon(t, s; \theta)\|_H^2 \leq e^{[L_G + 2\lambda](t-s)}\mathbb{E}\|\xi - \theta\|_H^2 \quad \forall 0 \leq s \leq t \leq T.$$

- $\mathcal{C}_b(H)$: the set of bounded continuous function from H to $\mathbb{R}$.

$$P_t^\epsilon \varphi : H \to \mathbb{R}; \quad (P_t^\epsilon \varphi)(\xi) := \mathbb{E}[\varphi(u_\epsilon(t; \xi))], \quad \xi \in H, \varphi \in \mathcal{C}_b(H),$$

where $u_\epsilon(t; \xi), t \geq 0$ is the unique strong solution to (8), $\xi \in H$.

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$$\mathbb{E}\|u_\epsilon(t, s; \xi) - u_\epsilon(t, s; \theta)\|_H^2 \leq e^{[L_G + 2\lambda](t-s)}\mathbb{E}\|\xi - \theta\|_H^2 \quad \forall 0 \leq s \leq t \leq T.$$

• $\mathcal{C}_b(H)$: the set of bounded continuous function from H to $\mathbb{R}$.

$$P_t^\epsilon \varphi : H \to \mathbb{R}; \quad (P_t^\epsilon \varphi)(\xi) := \mathbb{E}[\varphi(u_\epsilon(t; \xi))], \quad \xi \in H, \varphi \in \mathcal{C}_b(H),$$

where $u_\epsilon(t; \xi), t \geq 0$ is the unique strong solution to (8), $\xi \in H$.

• Standard arguments $\Longrightarrow$

$$P_{s,t}^\epsilon = P_{0,t-s}^\epsilon \quad \forall 0 \leq s \leq t < \infty \text{ and } P_t^\epsilon := P_{0,t}^\epsilon,$$

$$\mathbb{E}[\varphi(u_\epsilon(t+s; \eta))|\mathcal{F}_t] = (P_s^\epsilon \varphi)(u_\epsilon(t; \eta)) \quad \forall \varphi \in \mathcal{C}_b(H), \forall \eta \in H, \forall t, s > 0,$$

- Denote by $P^{\epsilon}_{s,t}\varphi(\xi) := \mathbb{E}[\varphi(u_\epsilon(t,s;\xi))], \quad \xi \in H.$

$$(P^{\epsilon}_{t+s}\varphi)(\eta) = (P^{\epsilon}_{t}(P^{\epsilon}_{s}\varphi))(\eta) \quad \forall \varphi \in \mathcal{C}_b(H), \forall \eta \in H, \forall t,s > 0.$$

Solution of (8) defines a homogeneous Feller–Markov process.

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Solution of (8) defines a homogeneous Feller–Markov process.

$$K_\psi = \{h \in H, \quad h \geq \psi\}$$

- Let $\eta \in K_\psi$ and $u(t;\eta) \in K_\psi, t \geq 0$ be the unique strong solution to (6).

$$\mathbb{E}\|u(t,s;\xi) - u(t,s;\theta)\|^2_H \leq e^{[L_G + 2\lambda_T](t-s)}\mathbb{E}\|\xi - \theta\|^2_H \quad \forall 0 \leq s \leq t \leq T,$$

$[u(t,s;\xi)]_{t\geq s\geq 0}$ is the unique strong solution to (6) with $u(s) = \xi \in K_\psi$.

Semigroup for the obstacle problem

$\rightsquigarrow u(t;\eta), t \geq 0$ be the unique strong solution to (6) starting from $\eta \in K_\psi$.

$$P_t\varphi : K_\psi \to \mathbb{R}; \quad (P_t\varphi)(\eta) := \mathbb{E}[\varphi(u(t;\eta))], \quad \eta \in K_\psi \quad \forall \varphi \in \mathcal{C}_b(H).$$

[2] L. Zambotti. *A reflected stochastic heat equation as symmetric dynamics with respect to the 3-d Bessel bridge.* JFA, 180(1), 195-209, 2001.

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Lemma

$(P_t)_t$ is bounded on $\mathcal{C}_b(H)$, Markov-Feller and stochastically continuous semigroup on $\mathcal{C}_b(K_\psi)$. Moreover, for $\varphi \in \mathcal{C}_b(H), \eta \in K_\psi$, and $t, s \geq 0$:

$$\lim_{\epsilon \to 0}(P_t^\epsilon \varphi)(\eta) = (P_t\varphi)(\eta), \quad (P_{t+s}\varphi)(\eta) = (P_t(P_s\varphi))(\eta)$$

$$P_{t+s,t} = P_{s,0},\ P_{t,s}\varphi(\eta) := \mathbb{E}[\varphi(u(t,s;\eta))] \text{ for } 0 \leq s \leq t < \infty,\ P_t := P_{0,t}.$$

- Convergence in law for u_ϵ, cv of $(P_t^\epsilon)_\epsilon$ and [Lemma 1] [2].

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Definition

Let $(P_t)_t$ be a Markov semigroup on H.

- a probability measure ν is said to be invariant for P_t if:

$$\int_H P_t\varphi d\nu = \int_H \varphi d\nu, \quad \forall \varphi \in B_b(H) \text{ and } t \geq 0.$$

- Let ν be an invariant measure for P_t, ν is ergodic if:

$$\lim_{T\to+\infty} \frac{1}{T}\int_0^T P_t\varphi dt = \int_H \varphi(x)\mu(dx), \quad \forall \varphi \in L^2(H,\mu).$$

- P_t is strongly mixing if: $\lim_{t\to+\infty} P_t\varphi(x) = \int_H \varphi(x)\mu(dx)$ in $L^2(H,\mu)$.

Theorem

Under the above assumptions H_1-H_6:

1. *There exists an ergodic invariant measure μ for $(P_t)_t$ if $p > 2$.*
2. *Assuming $p = 2$ and there exists $\bar{\alpha} > 0$ such that*

$$(1-\delta)\alpha - C_D[\lambda + \frac{L_G(1+K^2)}{2K^2}]^+ \geq \bar{\alpha} \text{ for some } \delta \in]0,1[, \forall K > 0$$

then there exists an ergodic invariant measure μ for $(P_t)_t$.

Moreover, μ is concentrated in V satisfying $\int_H \|x\|_V^p \mu(dx) < +\infty$.

$$C_D > 0 : \quad \|u\|_H^2 \leq C_D \|u\|_V^2, \quad \forall u \in V,$$

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Assuming $\max(1, \frac{2d}{d+2}) < p < 2$ and $\lambda + \frac{L_G(1+K^2)}{2K^2} \leq 0$. Then there exists an invariant measure μ for $(P_t)_t$!

The set $\{\nu_n := \frac{1}{t_n}\int_0^{t_n} \mu_s^\eta ds;\ n \in \mathbb{N}^*\}$ is tight on H.

μ_s^η is the law of $u(s;\eta)$

$$\frac{1}{2}\mathbb{E}\|u(t)\|_H^2 + (1-\delta)\alpha\int_0^t \mathbb{E}\|u\|_V^p ds \leq \frac{1}{2}\|\eta\|_H^2 + C(p,f,\psi,L_G)t$$

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Let $B_R := \{v \in V : \|v\|_V \le R\}; R \in \mathbb{N}$. Let $t_n > 0, R > 0$.

$$\nu_n(B_R^c) = \frac{1}{t_n}\int_0^{t_n} P(\|u(s;\eta)\|_V > R)ds \le \frac{1}{t_n R^p}\int_0^{t_n} \mathbb{E}\|u(s;\eta)\|_V^p ds.$$

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$\nu_n \rightharpoonup \mu \implies \int_H \|x\|_V^p \mu(dx) < +\infty \implies \mu$ is ergodic.

"Krein–Milman theorem".

Theorem

Assume moreover that $\frac{L_G}{2} + \lambda_T < 0$, *then there exists a* ***unique ergodic and strongly mixing invariant measure*** μ *and the following convergence to equilibrium holds.*

$$\left|P_t\varphi(x) - \int_H \varphi(y)\mu(dy)\right| \leq C\|\varphi\|_{1,\infty} e^{[\frac{L_G}{2}+\lambda_T]t}, \quad \forall \varphi \in \mathcal{C}_b^1(H). \tag{10}$$

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⇝ Difference of two ergodic invariant measures.

⇝ Continuity of P_t w.r.t. initial data.

$$du - [\mathrm{div}(|\nabla u|^{p-2}\nabla u) - u]dt + k1_{\{u=0\}}dt = fdt + ud\beta_t, \quad u \geq 0;$$

$$u_0 \in (L^2(D))^+, -f \in (L^2(D))^+ \qquad (p \geq 2)$$

Comments

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- Obstacle nonlinear problem with rough forcing e.g. $W(t)$? $\rightsquigarrow$ Open!
- Additive noise & invariant measures.

Thank you for your attention!

Yassine Tahraoui: Large deviations for an obstacle problem with T-monotone operator and multiplicative noise. (2023)
available on arXiv: https://arxiv.org/abs/2308.02206

—: Invariant measures and ergodicity for T-monotone problems with constraint and multiplicative noise. "preprint"