Lewy-Stampacchia's inequalities for obstacle problems

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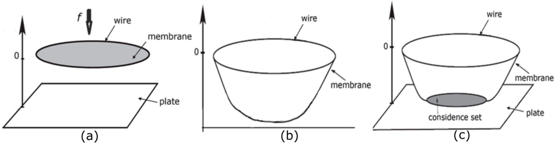
Obstacle problems

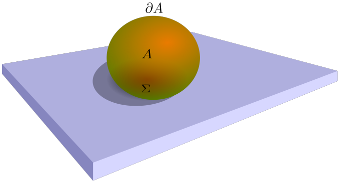
Signorini type problems



Apply a force on membrane

without plate (no obstacle)

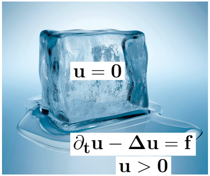
with plate (obstacle)

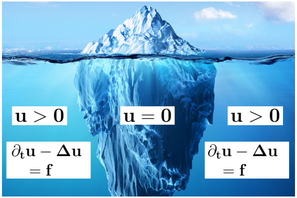


Equilibrium configuration: elastic body resting on a plane

Obstacle problems

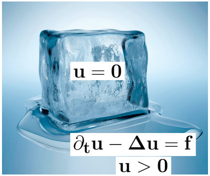
Stefan type problems

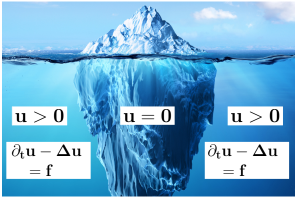




Obstacle problems

Stefan type problems





But also constraints in :

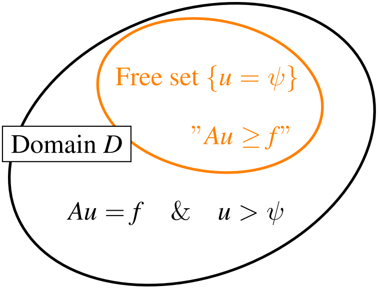
fluid flow in porous medium: contraint on the pressure to get or not a gas phase,

Model with constraints for vehicular traffic jams,

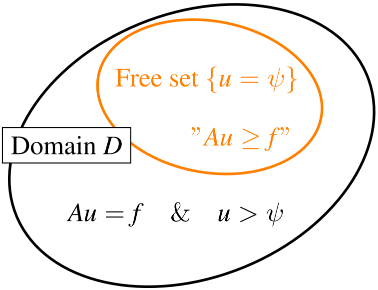
...

A free set: where/when the solution equals the constraint.

Obstacle problems $\quad D \subset \mathbb{R}^d \quad$ constraint ψ



A an operator to precise later



Penalization :

$$Au_\epsilon - \frac{1}{\epsilon}(u_\epsilon - \psi)^- = f$$

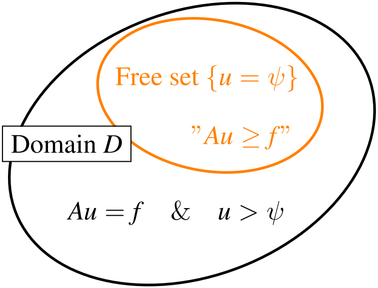
$$Au_\epsilon - f = \frac{1}{\epsilon}(u_\epsilon - \psi)^- \geq 0$$

$$\cdots \epsilon \to 0 \cdots$$

$$Au \geq f, \quad u \geq \psi$$

$$\& \quad (Au - f)(u - \psi) = 0$$

Obstacle problems $\quad D \subset \mathbb{R}^d \quad$ constraint ψ



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$$\cdots \epsilon \to 0 \cdots$$

$$Au \geq f, \quad u \geq \psi$$

$$\& \quad (Au - f)(u - \psi) = 0$$

$$\mu = Au - f \geq 0 \quad \& \quad \mu(u - \psi) = 0$$

Lewy-Stampacchia's inequalities: $\quad 0 \leq Au - f \ \leq \ (A\psi - f)^+$

Elliptic case : variational inequality

$$A : V = W_0^{1,p}(D) \to V' \qquad \text{or } V = W_0^{1,p}(D) \cap L^2(D) \text{ or } V = W_0^{1,p(x)}(D)$$

$$u \mapsto Au = -\mathrm{div}\Big(a(\cdot, u, \nabla u)\Big)$$

Leray-Lions pseudomonotone operator, + coercive, + growth conditions

$$\forall \epsilon > 0, \quad \exists u_\epsilon \in V = W_0^{1,p}(D), \quad Au_\epsilon - \frac{1}{\epsilon}\big(u_\epsilon - \psi\big)^- = f \quad \text{in } V'$$

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Test with $u_\epsilon - \psi$: $\qquad u_\epsilon \rightharpoonup u, \quad Au_\epsilon \rightharpoonup \chi, \quad \big(u_\epsilon - \psi\big)^- \to 0$

Elliptic case : variational inequality

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Test with $u_\epsilon - u$: $\quad \limsup \langle Au_\epsilon, u_\epsilon - u \rangle \leq 0$

$$\Rightarrow \qquad \lim \langle Au_\epsilon, u_\epsilon \rangle = \langle Au, u \rangle \quad \& \quad \chi = Au$$

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$$\mu_\epsilon := \frac{1}{\epsilon}\big(u_\epsilon - \psi\big)^- = Au_\epsilon - f \quad \to \quad \mu = Au - f \geq 0 \quad \text{in } V'$$

...✍... $\qquad u \geq \psi \quad \& \quad \langle \mu, u - \psi \rangle = 0.$

Elliptic case : variational inequality

$$A : V = W_0^{1,p}(D) \to V' \qquad \text{or } V = W_0^{1,p}(D) \cap L^2(D) \text{ or } V = W_0^{1,p(x)}(D)$$

$$u \mapsto Au = -\operatorname{div}\Big(a(\cdot, u, \nabla u)\Big)$$

Leray-Lions pseudomonotone operator, + coercive, + growth conditions

$$\exists (u, \mu) \in V \times V', \quad u \geq \psi, \quad \mu \geq 0, \qquad Au - \mu = f, \quad \langle \mu, u - \psi \rangle = 0,$$

i.e.

$$\exists u \in V, \quad u \geq \psi, \quad \forall v \in V, \qquad v \geq \psi \implies \langle Au, v - u \rangle \geq \langle f, v - u \rangle.$$

Rmk. $u \geq \psi$ & $\mu = Au - f$:

$$\text{"}\mu \geq 0,\ \langle \mu, u - \psi \rangle = 0\text{"} \Leftrightarrow \text{"}v \geq \psi \Rightarrow \langle \mu, v - u \rangle \geq 0\text{"}$$

Elliptic case : Lewy-Stampacchia's inequality

$$0 \leq \mu = Au - f \leq (A\psi - f)^+$$

Dual-order assumption:

$$A\psi - f = h^+ - h^-, \quad h^\pm \geq 0 \quad \text{in } V'.$$

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The "classical" case: A is strictly monotone [J.-F. Rodrigues *et al.*].

Set v solution to: $Av = f + h^+, \quad v \leq u$ (same type of problem)

Elliptic case : Lewy-Stampacchia's inequality

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Set v solution to: $Av = f + h^+, \quad v \leq u$ (same type of problem)

$\cdots$ ✍ $\cdots$ $i)\ v \geq \psi \ \cdots \quad ii)\ \langle Av - Au, v - u \rangle \leq 0 \ \cdots \quad iii)\ u = v \ \cdots$

$$iv)\ 0 \geq \lambda = Au - f - h^+, \qquad Au - f \leq h^+ \ !$$

Lewy-Stampacchia's inequalities

Dual-order assumption : $A\psi - f = h^+ - h^-$

$Au = -\text{div}[a(x, u, \nabla u)]$ pseudomonotone: (strictly monotone in ∇u only)

Lewy-Stampacchia's inequalities

Dual-order assumption : $A\psi - f = h^+ - h^-$

$Au = -\mathrm{div}[a(x,u,\nabla u)]$ pseudomonotone: (strictly monotone in ∇u only)

i. If $0 \leq h^+ \in L^2(D)$.

$$\mu_\epsilon = \frac{1}{\epsilon}(u_\epsilon - \psi)^- = Au_\epsilon - f$$

Test $\quad Au_\epsilon - A\psi - \frac{1}{\epsilon}(u_\epsilon - \psi)^- = h^- - h^+ \quad$ with $\quad \mu_\epsilon$

$$\frac{1}{\epsilon}\langle Au_\epsilon - A\psi, (u_\epsilon - \psi)^-\rangle + \frac{1}{\epsilon^2}\int_D |(u_\epsilon - \psi)^-|^2 dx$$

$$= \underbrace{-\frac{1}{\epsilon}\langle h^-, (u_\epsilon - \psi)^-\rangle}_{\leq 0} + \frac{1}{\epsilon}\int_D h^+ (u_\epsilon - \psi)^- dx \qquad \rightarrow \qquad \mu_\epsilon \text{ bounded in } L^2$$

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$$= \underbrace{-\frac{1}{\epsilon}\langle h^-, (u_\epsilon - \psi)^-\rangle}_{\leq 0} + \frac{1}{\epsilon}\int_D h^+(u_\epsilon - \psi)^- dx \qquad \to \qquad \mu_\epsilon \text{ bounded in } L^2$$

$$0 \leq \mu_\epsilon = h^+ - (h^+ - \mu_\epsilon) \leq h^+ + \underbrace{(h^+ - \mu_\epsilon)^-}_{\to 0 \quad (\textit{technical!})}$$

Lewy-Stampacchia's inequalities

Dual-order assumption : $A\psi - f = h^+ - h^-$

$Au = -\text{div}[a(x, u, \nabla u)]$ pseudomonotone: (strictly monotone in ∇u only)

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Lewy-Stampacchia's inequalities

Dual-order assumption :
$A\psi - f = h^+ - h^-$

$Au = -\mathrm{div}[a(x,u,\nabla u)]$ pseudomonotone: (strictly monotone in ∇u only)

i. If $0 \leq h_n^+ \in L^2(D)$.

$$u_n \geq \psi, \quad Au_n - f_n = \mu_n, \qquad 0 \leq Au_n - f_n \leq \left(A\psi - f_n\right)^+ = h_n^+.$$

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$$u_n \geq \psi, \quad Au_n - f_n = \mu_n, \qquad 0 \leq Au_n - f_n \leq \left(A\psi - f_n\right)^+ = h_n^+.$$

ii. $0 \leq h_n^+ \to h^+$ in V'. $\qquad 0 \leq \mu_n \leq h_n^+ \Rightarrow \underbrace{\mu_n \text{ bounded in } V'}_{\varphi\in V,\ \varphi=\varphi^+-\varphi^-\ !}.$

$$\mu_n \rightharpoonup \mu, \qquad Au - f = \mu, \qquad 0 \leq Au - f \leq \left(A\psi - f\right)^+ = h^+.$$

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$$\mu_n \rightharpoonup \mu, \qquad Au - f = \mu, \qquad 0 \leq Au - f \leq \left(A\psi - f\right)^+ = h^+.$$

$V = W_0^{1,p}(D)$ [A. Mokrane & F. Murat 98-04],

$V = W_0^{1,p(x)}(D)$ [A. Mokrane & G. V. 14]

+ conditions on $a(x,u,\vec{X}) - a(x,v,\vec{X})$.

$V = W_0^{1,p(x)}(D)$ [A. Mokrane, Y. Tahraoui & G. V. 18]

$$\widetilde{a}(x,u,\nabla u) = a(x,\max(u,\psi),\nabla u)$$

& bilateral problem.

Parabolic case. I. penalization $V = L^2(D) \cap W_0^{1,p}(D)$

$$\forall \epsilon > 0, \quad \exists u_\epsilon \in W(0,T) = \{u \in L^p(0,T,V),\ \partial_t u \in L^{p'}(0,T,V')\},$$

$$\partial_t u_\epsilon \underbrace{-\mathrm{div}\Big[a\big(t,x,\max(\psi,u_\epsilon),\nabla u_\epsilon\big)\Big]}_{\widetilde{A}(u_\epsilon)} - \frac{1}{\epsilon}\left[\big(u_\epsilon - \psi\big)^-\right]^{q-1} = f$$

$$q = \min(2,p)$$

$$\forall \epsilon > 0, \quad \exists u_\epsilon \in W(0,T) = \{u \in L^p(0,T,V),\ \partial_t u \in L^{p'}(0,T,V')\},$$

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$$q = \min(2,p)$$

Test with $u_\epsilon - \psi$

$$\|u_\epsilon\|_{C([0,T],L^2)} + \|u_\epsilon\|_{L^p(0,T,V)} + \frac{1}{\epsilon^{1/q}}\|\big(u_\epsilon - \psi\big)^-\|_{L^q(Q)} \leq C$$

$$u_\epsilon \rightharpoonup u \geq \psi \qquad \widetilde{A}u_\epsilon \rightharpoonup \chi$$

$$L^p(0,T,V)\ \&\ L^\infty(0,T,L^2) - * \qquad L^{p'}(0,T,V')$$

Parabolic case. I. penalization $V = L^2(D) \cap W_0^{1,p}(D)$

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Test with $u_\epsilon - \psi$

$$\|u_\epsilon\|_{C([0,T],L^2)} + \|u_\epsilon\|_{L^p(0,T,V)} + \frac{1}{\epsilon^{1/q}}\|\big(u_\epsilon - \psi\big)^-\|_{L^q(Q)} \leq C$$

$u_\epsilon \rightharpoonup u \geq \psi$ in $L^p(0,T,V)$ & $L^\infty(0,T,L^2) - *$

$\widetilde{A}u_\epsilon \rightharpoonup \chi$ in $L^{p'}(0,T,V')$

A three-terms equality

$$\partial_t u_\epsilon = \underbrace{f - \widetilde{A}u_\epsilon}_{bounded} + \underbrace{\frac{1}{\epsilon}\left[\big(u_\epsilon - \psi\big)^-\right]^{q-1}}_{\mu_\epsilon ???}$$

Dual-order assumption:

$\partial_t \psi + A\psi - f = h^+ - h^-, \quad h^\pm \geq 0 \quad \text{in } L^{p'}(0,T,V')$

Test with $-(u_\epsilon - \psi)^-$:

$$\left\langle \partial_t(u_\epsilon - \psi), -(u_\epsilon - \psi)^- \right\rangle - \left\langle Au_\epsilon - A\psi, (u_\epsilon - \psi)^- \right\rangle + \frac{1}{\epsilon}\|(u_\epsilon - \psi)^-\|^q_{L^q(D)}$$

$$= \left\langle \partial_t\psi + A\psi - f, (u_\epsilon - \psi)^- \right\rangle \cdots \leq \cdots \left\langle h^+, (u_\epsilon - \psi)^- \right\rangle$$

I. penalization

$\partial_t u_\epsilon + Au_\epsilon - \frac{1}{\epsilon}[(u_\epsilon - \psi)^-]^{q-1} = f$

Dual-order assumption:

$\partial_t \psi + A\psi - f = h^+ - h^-, \quad h^\pm \geq 0 \quad \text{in } L^{p'}(0,T,V')$

Test with $-(u_\epsilon - \psi)^-$:

$$\left\langle \partial_t(u_\epsilon - \psi), -(u_\epsilon - \psi)^- \right\rangle - \left\langle Au_\epsilon - A\psi, (u_\epsilon - \psi)^- \right\rangle + \frac{1}{\epsilon}\|(u_\epsilon - \psi)^-\|^q_{L^q(D)}$$
$$= \left\langle \partial_t \psi + A\psi - f, (u_\epsilon - \psi)^- \right\rangle \leq \int_D h^+ (u_\epsilon - \psi)^- dx.$$

i. $0 \leq h^+ \in L^{q'}(Q)$.

$$\frac{1}{\epsilon}\|(u_\epsilon - \psi)^-\|^{q-1}_{L^q(Q)} \leq C \quad \Longrightarrow \quad \partial_t u_\epsilon \quad \text{bounded} \quad L^{p'}(0,T,V')$$

& Aubin-Lions-Simon

Dual-order assumption:

$$\partial_t \psi + A\psi - f = h^+ - h^-, \quad h^\pm \geq 0 \quad \text{in } L^{p'}(0,T,V')$$

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& Aubin-Lions-Simon

$$\int_Q \left| [\widetilde{a}(\cdot, u_\epsilon, \nabla u_\epsilon) - \widetilde{a}(\cdot, \psi, \nabla\psi)]\nabla(u_\epsilon - \psi)^- \right| \leq C(\|h^+\|_{L^{q'}(Q)})\epsilon^{1/q}$$

$\Longrightarrow$ Minty + conv. in measure of gradient

$$\exists (u,\mu) \in \cdots, \quad u \geq \psi, \quad \mu \geq 0, \quad \mu(u-\psi) = 0, \quad \partial_t u + Au - f = \mu \geq 0$$

Dual-order assumption:

$$\partial_t \psi + A\psi - f = h^+ - h^-, \quad h^\pm \geq 0 \quad \text{in } L^{p'}(0,T,V')$$

i. $0 \leq h^+ \in L^{q'}(Q)$.

$$\exists (u,\mu) \in \cdots, \quad u \geq \psi, \quad \mu \geq 0, \quad \mu(u-\psi) = 0, \quad \partial_t u + Au - f = \mu \geq 0$$

$$\ldots ✍ \ldots \left(h^+ - \frac{1}{\epsilon}[(u_\epsilon - \psi)^-]^{q-1}\right)^- \to 0 \qquad (\textit{technical!})$$

$$0 \leq \mu = \partial_t u + Au - f \leq \left(\partial_t \psi + A\psi - f\right)^+ = h^+$$

II. Lewy-Stampacchia $\partial_t u_\epsilon + Au_\epsilon - \frac{1}{\epsilon}[(u_\epsilon - \psi)^-]^{q-1} = f$

$\partial_t \psi + A\psi - f = h^+ - h^-, \quad h^\pm \geq 0 \quad \text{in } L^{p'}(0,T,V')$

i. $0 \leq h_n^+ \in L^{q'}(Q)$. $\quad \exists (u_n, \mu_n) \in W(0,T) \times L^{p'}(0,T,V')$,

$$u_n \geq \psi, \quad 0 \leq \partial_t u_n + Au_n - f_n = \mu_n \leq h_n^+.$$

ii. $0 \leq h_n^+ \to h^+$ in $L^{p'}(0,T,V')$.

$$u_n \quad \text{bounded in} \quad C([0,T],L^2(D)) \quad \text{and} \quad L^p(0,T,V)$$
$$\Rightarrow Au_n \quad \text{bounded in} \quad L^{p'}(0,T,V')$$

$$0 \leq \mu_n \leq h_n^+ \quad \Rightarrow \quad \partial_t u_n \quad \text{bounded in} \quad L^{p'}(0,T,V').$$

$\varphi \in L^p(0,T,V),\ \varphi = \varphi^+ - \varphi^-$!

... the result holds.

Remarks and technical results

A monotone: [F. Donati 82]

A pseudomonotone: [O. Guibé, A. Mokrane, Y. Tahraoui, G. V. 18]

$D \subset \mathbb{R}^N$ bounded Lipschitz domain

$$\left\{ u \in L^p(0,T; W^{1,p}_{\cancel{0}}(D)),\ \partial_t u \in L^{p'}(0,T; W^{-1,p'}(D)) \right\} \hookrightarrow C([0,T], L^2(D)).$$

Chain rule with $\langle \partial_t u, \Psi(t,x,u) \rangle$.

III . Parabolic problem with L^1 data $\mu = \partial_t u - \mathrm{div}[a(t,x,u,\nabla u)] - f$

$$f \in L^1(Q_T) + L^{p'}(0,T,V') \qquad u_0 \in L^1(D)$$

Dual-order assumption: $\quad f - \partial_t \psi - A(\psi) = g^+ - g^-,$

$$0 \leq g^{\pm} \in L^1(Q) + L^{p'}(0,T,W^{-1,p'}(D)).$$

Definition (Entropy solution)

$$u \in L^\infty\left(0,T,L^1(D)\right), \quad u \geq \psi, \quad \forall k>0,\ T_k(u) \in L^p(0,T,W_0^{1,p}(D)).$$

$$V_1^p(0,T) = \{u \in L^p(0,T,W_0^{1,p}(D)),\ \partial_t u \in L^{p'}(0,T,W^{-1,p'}(D)) + L^1(Q)\}.$$

$\forall k>0,\ v \in V_1^p(0,T) \cap L^\infty(Q),\ v \geq \psi \Rightarrow$ $\qquad (\tilde{T}_k' = T_k)$

$$\int_D \tilde{T}_k(v(0)-u_0)dx + \int_0^T \langle \partial_t v - f, T_k(v-u)\rangle dt + \int_Q a(t,x,u,\nabla u)\nabla T_k(v-u)dxdt \geq 0$$

III . Parabolic problem with L^1 data $\mu = \partial_t u - \text{div}[a(t,x,u,\nabla u)] - f$

$$f \in L^1(Q_T) + L^{p'}(0,T,V') \qquad u_0 \in L^1(D)$$

Dual-order assumption: $$f - \partial_t \psi - A(\psi) = g^+ - g^-,$$

$$0 \leq g^\pm \in L^1(Q) + L^{p'}(0,T,W^{-1,p'}(D)).$$

Exists: $(g_n^\pm) \subset L^{p'}(0,T,W_0^{-1,p'}(D))^+$

$$g_n^\pm \to g^\pm \quad \text{in} \quad L^1(Q) + L^{p'}(0,T,W^{-1,p'}(D))$$

$$\Rightarrow \exists (u_n, \mu_n) \in \cdots \qquad \cdots 0 \leq \mu_n \leq g_n^+$$

Pb.:

inequality and not equality

$$L^1(Q) + L^{p'}(0,T,W^{-1,p'}(D)) \not\subseteq \left(L^\infty(Q) \cap L^p(0,T,W_0^{1,p}(D))\right)'$$

Definition (Renormalized Lewy-Stampacchia's inequalities)

$F^i \in L^1(Q), \quad G^i \in L^{p'}(Q)^d$ (i=1,2) $\quad u_0 \in L^1(D)$.

$$\boxed{F^1 - \operatorname{div}\left(G^1\right) \leq \frac{\partial u}{\partial t} - \operatorname{div}(a(\cdot, u, \nabla u)) \leq F^2 - \operatorname{div}\left(G^2\right) \text{ with } u(t=0) = u_0}$$

means :

$$u \in L^\infty\left(0, T, L^1(D)\right), \quad \forall k > 0,\ T_k(u) \in L^p(0, T, W_0^{1,p}(D)),$$

For all suitable S and φ,
$$\lim_{n \to +\infty} \frac{1}{n} \int_{\{|u|<n\}} a(\cdot, u, \nabla u)\nabla u dxdt = 0$$

$$\int_Q F^1 S'(u)\varphi + G^1 \cdot \nabla\left(S'(u)\varphi\right) dxdt$$
$$\leq -\int_Q \frac{\partial \varphi}{\partial t} S(u) dxdt - \int_D \varphi(0,x) S\left(u_0\right) dx + \int_Q a(t,x,u,\nabla u)\nabla\left(S'(u)\varphi\right) dxdt$$
$$\leq \int_Q F^2 S'(u)\varphi + G^2 \cdot \nabla\left(S'(u)\varphi\right) dxdt.$$

Theorem (Renormalized techniques O. Guibé, Y. Tahraoui, G. V. (sub.))

Compactness & stability for $F_n^i \to F^i$ in $L^1(Q)$, $G_n^i \to G^i$ in $L^{p'}(Q)$ and $u_{0,n} \to u_0$ in $L^1(Q)$.

Stochastic case. I. penalization $V = L^2(D) \cap W_0^{1,p}(D)$

$Au = -\text{div}[a(\omega, t, x, u, \nabla u)]$ monotone (+ coercive + growth cond.)

$f \in L^{p'}_{\mathcal{P}}(\Omega_T, V'),$ $(\Omega_T = \Omega \times (0,T),$ $\mathcal{P}$ for predictable)

$$\forall \epsilon > 0, \quad \exists u_\epsilon \in L^2\big(\Omega, C([0,T], L^2(D))\big) \cap L^p_{\mathcal{P}}(\Omega_T, V), \qquad u(0) = u_0,$$

$$\partial_t \Big[u_\epsilon - \int_0^\cdot \underbrace{G(\cdot, \max(u_\epsilon, \psi))}_{\widetilde{G}(\cdot,u)} dW \Big] \underbrace{-\text{div}\Big[a\big(\cdot, u_\epsilon, \nabla u_\epsilon\big)\Big]}_{A(u_\epsilon)} - \frac{1}{\epsilon}\Big[\big(u_\epsilon - \psi\big)^-\Big]^{q-1} = f$$

$$q = \min(2,p)$$

$Au = -\mathrm{div}[a(\omega, t, x, u, \nabla u)]$ monotone (+ coercive + growth cond.)

$f \in L^{p'}_{\mathcal{P}}(\Omega_T, V')$, $(\Omega_T = \Omega \times (0,T)$, $\mathcal{P}$ for predictable)

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$q = \min(2, p)$

Itô energy:

$$\mathbb{E}\|u_\epsilon\|^2_{C([0,T],L^2)} + \mathbb{E}\|u_\epsilon\|^p_{L^p(0,T,V)} + \tfrac{1}{\epsilon}\mathbb{E}\|\big(u_\epsilon - \psi\big)^-\|^q_{L^q(Q)} \le C$$

$u_\epsilon \rightharpoonup u \geq \psi$ $\qquad$ $\widetilde{A}u_\epsilon \rightharpoonup \chi$

$L^p_{\mathcal{P}}(\Omega_T, V)$ & $L^2_w(\Omega, L^\infty(0,T,L^2)) - *$ $\qquad$ $L^{p'}_{\mathcal{P}}(\Omega_T, V')$

Dual-order assumption :

$$\partial_t\left(\psi - \int_0^\cdot G(\psi)dW\right) + A\psi - f = h^+ - h^-, \quad h^\pm \geq 0 \quad \text{in } L^{p'}_{\mathcal{P}}(0,T,V')$$

i. h^+ regular : Itô formula to simulate $\|(u_\epsilon - \psi)^-\|^2_{L^2}$

$$\mu_\epsilon = -\frac{1}{\epsilon}[(u_\epsilon - \psi)^-]^{q-1} \quad \text{bounded in} \quad L^{q'}_{\mathcal{P}}(\Omega_T, L^{q'}(D))$$

$$\Longrightarrow (u_\epsilon) \quad \text{Cauchy} \quad \text{in} \quad L^2(\Omega, C([0,T], L^2(D))$$

$$\Longrightarrow \chi = Au \quad \text{Minty.}$$

$$\exists u \in L^p_{\mathcal{P}}(\Omega_T, V) \cap L^2(\Omega, C([0,T], L^2)), \quad \mu \in L^{q'}_{\mathcal{P}}(\Omega_T, L^{q'}(D)), \quad u(0) = u_0,$$

$$\mu \geq 0, \quad u \geq \psi, \quad \mu(u-\psi) = 0, \quad du + [Au - \mu]dt = fdt + G(u)dW.$$

II. Lewy-Stampacchia

$$\exists u \in L^p_{\mathcal{P}}(\Omega_T, V) \cap L^2(\Omega, C([0,T], L^2)), \quad \mu \in L^{q'}_{\mathcal{P}}(\Omega_T, L^{q'}(D)), \quad u(0) = u_0,$$
$$\mu \geq 0, \quad u \geq \psi, \quad \mu(u-\psi) = 0, \quad du + [Au - \mu]dt = fdt + G(u)dW.$$

Similarly

$$\exists v \in L^p_{\mathcal{P}}(\Omega_T, V) \cap L^2(\Omega, C([0,T], L^2)), \quad \lambda \in L^{q'}_{\mathcal{P}}(\Omega_T, L^{q'}(D)), \quad v(0) = u_0,$$
$$\lambda \leq 0, \quad v \leq u, \quad \lambda(v-u) = 0, \quad dv + [Av - \lambda]dt = (f + h^+)dt + G(v)dW.$$

II. Lewy-Stampacchia

$$\exists u \in L^p_{\mathcal{P}}(\Omega_T, V) \cap L^2(\Omega, C([0,T], L^2)), \quad \mu \in L^{q'}_{\mathcal{P}}(\Omega_T, L^{q'}(D)), \quad u(0) = u_0,$$

$$\mu \geq 0, \quad u \geq \psi, \quad \mu(u-\psi) = 0, \quad du + [Au - \mu]dt = fdt + G(u)dW.$$

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$$\lambda \leq 0, \quad v \leq u, \quad \lambda(v-u) = 0, \quad dv + [Av - \lambda]dt = (f + h^+)dt + G(v)dW.$$

Uniqueness method: $u = v$

$$0 \leq \partial_t\left(u - \int_0^\cdot G(u)dW\right) + Au - f \leq h^+.$$

h^+ regular

II. Lewy-Stampacchia

i. $0 \leq h^+ \in L^{q'}_{\mathcal{P}}(\Omega_T, L^{q'}(D))$. $\exists (u\ \ , \mu\ \ ) \in \cdots$,

$$0 \leq \partial_t \Big(u\ \ - \int_0^{\cdot} G(u\ \ ) dW \Big) + Au\ \ - f\ \ = \mu\ \ \leq h^+.$$

II. Lewy-Stampacchia

i. $0 \leq h_n^+ \in L^{q'}_{\mathcal{P}}(\Omega_T, L^{q'}(D))$. $\exists (u_n, \mu_n) \in \cdots$,

$$0 \leq \partial_t \Big(u_n - \int_0^\cdot G(u_n)dW \Big) + Au_n - f_n = \mu_n \leq h_n^+.$$

ii. $0 \leq h_n^+ \to h^+$ in $L^{p'}_{\mathcal{P}}(\Omega_T, V')$.

II. Lewy-Stampacchia

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ii. $0 \leq h_n^+ \to h^+$ in $L^{p'}_{\mathcal{P}}(\Omega_T, V')$.

$$0 \leq \mu_n \leq h_n^+ \quad \text{bounded in} \quad L^{p'}_{\mathcal{P}}(\Omega_T, V')$$

$$\Longrightarrow (u_n) \quad \text{Cauchy} \quad \text{in} \quad L^2(\Omega, C([0,T], L^2(D)))$$

$$\Longrightarrow Au_n \rightharpoonup Au \quad \text{Minty}.$$

... the result holds [Y. Tahraoui, G. V. 21].

Rmk. Need of an equation ($i.e.$ μ) to apply Itô's formula

The pseudomonotone case

SPDE with constraints and a pseudomonotone operator

Compactness methods in stochastic: Prokhorov and Skorokhod

16:00 Niklas Sapountzoglou,

& Yassine Tahraoui, GV, Aleksandra Zimmermann

The hyperbolic case

Hyperbolic SPDE with constraints (I. Biswas, Y. Tahraoui, G. V. 23)

$$du - \operatorname{div} f(t,x,u) + \Lambda = g(t,x,u) + G(t,u)dW, \quad 0 \le u \le \psi$$

with suitable assumptions.

Definition (Entropy obstacle formulation)

A solution is $(u, \Lambda_1, \Lambda_2) \in L^2_{\mathcal{P}}(\Omega_T, L^2(\mathbb{R}^d))^3$, s.t. $\Lambda = \Lambda_1 - \Lambda_2$,

$$0 \le u \le \psi, \quad 0 \le \Lambda_1, \Lambda_2, \quad \Lambda_1(u-\psi) = 0, \quad \Lambda_2 u = 0,$$

$$\int_{Q_T} \Big[\eta(u-k)\,\partial_t\varphi - \int_k^u \eta'(\tau-k)\partial_u f(\cdot,\tau)\,d\tau\nabla\varphi + [g(\cdot,u)-\Lambda]\varphi\eta'(u-k)\Big]\,dx\,dt$$
$$+ \sum_{\ell\ge1}\int_{Q_T}\sigma_\ell(\cdot,u)\eta'(u-k)\varphi\,d\beta_\ell(t)\,dx + \frac{1}{2}\int_{Q_T}\sum_{\ell\ge1}\sigma_\ell(\cdot,u)^2\eta''(u-k)\varphi\,dx\,dt$$
$$+ \int_{\mathbb{R}^d}\eta(u_0-k)\varphi(0)\,dx \ge 0.$$

$k \in \mathbb{R}$, (η, f) regular convex entropy pair, $\varphi \in D^+([0,T[\times\mathbb{R}^d)$

The hyperbolic case

Hyperbolic SPDE with constraints (I. Biswas, Y. Tahraoui, G. V. 23)

$$du - \operatorname{div} f(t,x,u) + \Lambda = g(t,x,u) + G(t,u)dW, \quad 0 \le u \le \psi$$

with suitable assumptions. $\Lambda = \Lambda_1 - \Lambda_2$

Theorem (Lewy-Stampacchia's inequalities)

$\exists!(u,\Lambda)$ *solution s.t.*

$$0 \le \Lambda^1 \le \Big(-\partial_t\big(\psi - \int_0^\cdot G(\cdot,\psi)dW\big) + divf(\cdot,\psi) + g(\cdot,\psi)\Big)^+$$

$$0 \le \Lambda^2 \le g(\cdot,0)^- \qquad \text{in } L^2(\Omega \times Q_T)$$

The hyperbolic case

Hyperbolic SPDE with constraints (I. Biswas, Y. Tahraoui, G. V. 23)

$$du - \operatorname{div} f(t,x,u) + \Lambda = g(t,x,u) + G(t,u)dW, \quad 0 \le u \le \psi$$

with suitable assumptions. $\Lambda = \Lambda_1 - \Lambda_2$

Uniqueness:
Kruzkhov's doubling variable (importance of having Lagrange multipliers),
Kato's inequality (properties of the Λs).

Existence:
Parabolic obstacle problem with artificial viscosity and Lagrange multiplier (importance of having an equation for Itô's formula),
a priori estimates thanks to parabolic Lewy-Stampacchia's inequalities,
Young-measures.