Explicit Euler scheme: a new way of proof for the existence of solutions for singular parabolic SPDEs

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Modèles stochastiques appliqués à la mécanique : aspects théoriques et numériques

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1. Singular model of equations
   - Numerical simulations
   - SDE Weak solutions

2. Sewing approach and Tamed Euler scheme
   - Sewing approach
   - Tamed Euler scheme
   - Strong existence and uniqueness of the SDE

3. Tamed Euler scheme for SPDE
   - Finite differences
   - Controlling the error
1. **Singular model of equations**
   - Numerical simulations
   - SDE Weak solutions

2. **Sewing approach and Tamed Euler scheme**
   - Sewing approach
   - Tamed Euler scheme
   - Strong existence and uniqueness of the SDE

3. **Tamed Euler scheme for SPDE**
   - Finite differences
   - Controlling the error
Consider the stochastic reaction-diffusion

\[ \partial_t u(t, x) = \frac{1}{2} \partial_{xx} u(t, x) + b(u(t, x)) + \xi \]  

(*)

where \( \xi \) is a space-time white noise and \( b \) is a (one-sided) Lipschitz function, i.e. polynomial-like function with negative sign of leading coefficient.
Consider the stochastic reaction-diffusion

$$\partial_t u(t, x) = \frac{1}{2} \partial_{xx}^2 u(t, x) + b(u(t, x)) + \xi$$  \hspace{1cm} (*)

where $\xi$ is a space-time white noise and $b$ is a (one-sided) Lipschitz function, i.e. polynomial-like function with negative sign of leading coefficient.

The existence and uniqueness of solutions are quite well-understood, but these results do not cover the natural cases of discontinuous drifts or reflected diffusions which are expected in physical models.
Consider again the stochastic reaction-diffusion

\[ \partial_t u(t, x) = \frac{1}{2} \partial_{xx} u(t, x) + b(u(t, x)) + \xi \]  

(\star)

where \( \xi \) is a space-time white noise, but where \( b \) is a generalized function in Besov space \( \mathcal{B}_{p,\infty}^{\gamma}(\mathbb{R}^d) \).

In dimension \( d = 1 \), Athreya, Butkovsky, Lê, Mytnik [ABLM21] proved that there exists a unique strong solution whenever \( b \) has some Hölder regularity (precisely Besov regularity \( \gamma - 1/p \geq -1 \), \( \gamma > -1 \) and \( p \in [1, \infty] \)).
SPDE with singular drift

We can treat the following case

$$\partial_t u(t, x) = \frac{1}{2} \partial_{xx}^2 u(t, x) + k1_{u=0}(u(t, x)) + W_t$$
SPDE with singular drift

We can treat the following case

$$\partial_t u(t, x) = \frac{1}{2} \partial_{xx}^2 u(t, x) + k1_{u=0}(u(t, x)) + W_t$$

So a new way of proof for SPDE with singular drift, reflection, penalization.
We can treat the following case

\[ \partial_t u(t, x) = \frac{1}{2} \partial_{xx}^2 u(t, x) + k\mathbf{1}_{u=0}(u(t, x)) + \mathcal{W}_t \]

So a new way of proof for SPDE with singular drift, reflection, penalization. Furthermore, without monotonic behavior, or (one-sided) Lipschitz condition.
Euler scheme

For the SPDE

\[ dX_t = AX_t dt + b(X_t) dt + dW_t \]

consider the Euler scheme with time step \( h = T/N \)

\[ X^{n+1} = X^n + A(X^n)h + b(X^n)h + \Delta W^{n+1} \]

then the strong order is given by

\[ \mathbb{E} \left[ \sup_{0 \leq n \leq N} \| X_{nh} - X^n \|^2 \right] \leq C_T h. \]
Euler scheme: rate of convergence

\[ \mathbb{E} \left[ \sup_{0 \leq n \leq N} \| X_{nh} - X^n \|^2 \right] \leq C_T h. \]

The usual way of proof is to obtain the bound of moments

\[ \mathbb{E} \left[ \sup_{0 \leq n \leq N} \| X^n \|^p \right] \leq C_T, \]

and it is expected that the order of convergence is given by the time regularity of the noise based on the control of

\[ \mathbb{E} \left[ \sup_{0 \leq t, s \leq T, |t-s| < h} \| W_t - W_s \|^2 \right] \leq C_T h \quad \text{and} \quad \mathbb{E} \left[ \sup_{0 \leq t \leq T} \| W_t \|^p \right] \leq C_T, \]

There are many way to obtain the same regularity/moment bounds for the Euler scheme, but it NEEDS an implicit treatment of the differential operator (see also exponential scheme, splitting, etc.). Otherwise it has been shown that the forward/explicit Euler scheme may not be convergent (for instance with superlinear growing of \( b \)).
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   - Finite differences
   - Controlling the error
Bounded drift

Consider

\[
dX_t = 1_{\{X_t > 0\}} dt + dB_t^H,
\]
\[
dX_t = 1^{(d)}_D (X_t) dt + dB_t^H.
\]

Then \( b \in C^0_0 / B^0_{\infty} \) leads to \( b^n(x) = \sqrt{\frac{n}{2\pi}} \int_0^x e^{-\frac{ny^2}{2}} dy \) and \( n = \lfloor h^{-1} \rfloor \).

Dirac drift

Consider

\[
dX_t = \delta_0(X_t) dt + dB_t^H.
\]

Then \( b \in C^{-1} / B^{d+d/p}_{p} \) leads to \( b^n(x) = \sqrt{\frac{n}{2\pi}} e^{-\frac{nx^2}{2}} \) and \( n = \lfloor h^{-\frac{1}{1+d}} \rfloor \).

Set \( h = 2^{-7}10^{-4} \) as reference value.
Indicator function in dimension 1 - exact algorithm

Figure: It works!
Indicator function in dimension 1 - \textbf{tamed} algorithm

\textbf{Figure}: It works quite well...
Dirac distribution in dimension 1 - naive algorithm

Figure: It does not work!
Dirac distribution in dimension - *tamed* algorithm

Figure: It works quite well...
Indicator function in dimension 1

![Graph showing approximate slope is 0.5](image)

**Figure:** Approximate slope is 0.5
Indicator function in dimension 2

Figure: Approximate slope is 0.5
Dirac $\delta$ function in dimension 1

Figure: Approximate slope is 0.25
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We study the equation

\[ dX_t = b(X_t)dt + dB^H_t, \]

when \( b \) is a distribution in some Hölder/Besov space and \( B^H \) is a fractional Brownian motion.

We look for solutions of the form

\[ X_t = X_0 + K_t + B^H_t, \]

where in case \( b \) is regular enough, \( K_t = \int_0^t b(X_s)ds \).

**F-fBm**

For a filtration \( \mathbb{F} \), we say that \( B^H \) is an \( \mathbb{F} \)-fBm if there exists an \( \mathbb{F} \)-Brownian motion \( W \) s.t. \( B^H_t = \int_0^t K_H(t, s) dW_s. \)
Weak solutions

\[ X_t = X_0 + \int_0^t b(X_s) \, ds + B_t^H, \quad t \in [0, T]. \]  

Definition

\((X_t)_{t \in [0, T]}, (B_t)_{t \in [0, T]}\) defined on some filtered probability space \((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})\) is a \textit{weak solution} of (*) if

- \(B^H\) is an \(\mathbb{F}\)-fBm;
- \(X\) is adapted to \(\mathbb{F}\);
- \(\exists (K_t)_{t \in [0, T]}\) such that, a.s.,

\[ X_t = X_0 + K_t + B_t^H, \quad \forall t \in [0, T]; \]

- \(\forall (b^k)_{k \in \mathbb{N}}\) smooth bounded functions converging to \(b\) in \(C^\alpha / B_p^\gamma\),

\[ \sup_{t \in [0, T]} \left| \int_0^t b^k(X_r) \, dr - K_t \right| \xrightarrow{\mathbb{P}} 0. \]
Consider $b$ is a generalized function in

Besov space $\mathcal{B}^{\gamma,p}_{p,\infty}(\mathbb{R}^d)$. 
Consider \( b \) is a generalized function in

\[
\text{Besov space } B^{\gamma}_{p, \infty} (\mathbb{R}^d).
\]

A few properties:

- \( B^{\gamma}_{p, \infty} (\mathbb{R}^d) \hookrightarrow C^{\gamma - d/p} (\mathbb{R}^d) \) (\( \gamma - d/p \) is the “regularity” of the space);
  
  You can think: \( \alpha = \gamma - d/p \) is the Hölder regularity.
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  You can think: $\alpha = \gamma - d/p$ is the Hölder regularity.
- For $\gamma - d/p = 0$, the space $\mathcal{B}_{p,\infty}^{\gamma}(\mathbb{R}^d)$ contains bounded functions;
Consider $b$ is a generalized function in

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  You can think: $\alpha = \gamma - d/p$ is the Hölder regularity.

- For $\gamma - d/p = 0$, the space $\mathcal{B}^{\gamma}_{p,\infty}(\mathbb{R}^d)$ contains bounded functions;

- $1_{\mathbb{R}_+} \in \mathcal{B}^{0}_{\infty,\infty}(\mathbb{R}^1)$. 
Consider $b$ is a generalized function in

Besov space $\mathcal{B}_{p,\infty}^\gamma(\mathbb{R}^d)$.

A few properties:

- $\mathcal{B}_{p,\infty}^\gamma(\mathbb{R}^d) \hookrightarrow C^{\gamma-d/p}(\mathbb{R}^d)$ ($\gamma - d/p$ is the “regularity” of the space);
  You can think: $\alpha = \gamma - d/p$ is the Hölder regularity.
- For $\gamma - d/p = 0$, the space $\mathcal{B}_{p,\infty}^\gamma(\mathbb{R}^d)$ contains bounded functions;
- $1_{\mathbb{R}^+} \in \mathcal{B}_{\infty,\infty}^0(\mathbb{R}^1)$.
- For $\gamma - d/p < 0$, the space $\mathcal{B}_{p,\infty}^\gamma(\mathbb{R}^d)$ contains genuine distributions;
Consider $b$ is a generalized function in

Besov space $\mathcal{B}^\gamma_{p,\infty}(\mathbb{R}^d)$.

A few properties:

- $\mathcal{B}^\gamma_{p,\infty}(\mathbb{R}^d) \hookrightarrow C^{\gamma-d/p}(\mathbb{R}^d)$ ($\gamma - d/p$ is the “regularity” of the space);
  You can think: $\alpha = \gamma - d/p$ is the Hölder regularity.
- For $\gamma - d/p = 0$, the space $\mathcal{B}^\gamma_{p,\infty}(\mathbb{R}^d)$ contains bounded functions;
- $1_{\mathbb{R}^+} \in \mathcal{B}^0_{\infty,\infty}(\mathbb{R}^1)$.
- For $\gamma - d/p < 0$, the space $\mathcal{B}^\gamma_{p,\infty}(\mathbb{R}^d)$ contains genuine distributions;
- $\delta_0 \in \mathcal{B}^0_{1,\infty}(\mathbb{R})$ (or $\mathcal{B}^{-d+d/p}_{p,\infty}(\mathbb{R}^d)$ in dimension $d$).
Weak and strong existence

**Theorem ([Anzeletti-Richard-Tanré ’21], [G.-Haress-Richard.’22])**

Let $\gamma \in \mathbb{R}$, $p \in [1, \infty]$, $b \in C^\alpha / B_p^\gamma$.

*weak* Assume that

$$\alpha = \gamma - \frac{d}{p} > \frac{1}{2} - \frac{1}{2H}. $$

Then there exists a weak solution $X$ s.t. $X - B^H \in C_{[0,T]}^\kappa (L^m)$, for all $m \geq 2$,

$$\forall \kappa \in (0, 1 + H\alpha] \setminus \{1\},$$

$$\forall \kappa \in \left(0, 1 + H \left(\gamma - \frac{d}{p}\right) \right] \setminus \{1\}.$$

*strong* Assume that

$$H < \frac{1}{2}, \quad \gamma - \frac{d}{p} < 0 \text{ and } \gamma - \frac{d}{p} > 1 - \frac{1}{2H}. $$

Then there exists a strong solution $X$ to (*) such that $X - B \in C_{[0,T]}^{\frac{1}{2} + H} (L^m)$ for any $m \geq 2$. Besides, pathwise uniqueness holds in the class of all solutions $X$ such that $X - B \in C_{[0,T]}^{\frac{1}{2} + H} (L^2)$. 

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New way of proof
Reflection?

Example

If \( b = \delta_0 \) (\( b \in C^{-1}/B^0_1 \)) and \( d = 1 \), one must choose \( H < \frac{1}{3} \).

Then \( X - B^H \) has Hölder regularity \( 1 - H > H \), hence \( X \) is not reflected.
We conclude the introduction by commenting on the needs to treat SDE first. It is known that for each fixed space point, the free stochastic heat equation (that is, equation (⋆) with $b = 0$) behaves “qualitatively” like a fractional Brownian motion (fBM) with the Hurst parameter $H = 1/4$. 
SPDE with singular drift

We conclude the introduction by commenting on the needs to treat SDE first.

It is known that for each fixed space point, the free stochastic heat equation (that is, equation (⋆) with \( b = 0 \)) behaves “qualitatively” like a fractional Brownian motion (fBM) with the Hurst parameter \( H = 1/4 \).

Indeed, the solution of

\[
\partial_t u(t, x) = \frac{1}{2} \partial_{xx} u(t, x) + \dot{W}(t, x)
\]

is given by the well-known stochastic convolution

\[
Z(t, x) = e^{\frac{t}{2} \Delta} u_0(x) + \int_0^t e^{\frac{t-s}{2} \Delta} dW(s, x)
\]

which has regularity \( C^{\frac{1}{4}-} \) in time and \( C^{\frac{1}{2}-} \) in space.
To solve
\[ \partial_t u(t, x) = \frac{1}{2} \partial_{xx}^2 u(t, x) + b(u(t, x)) + \dot{W}(t, x). \] (*)

Denote \( Y(t, x) = u(t, x) - Z(t, x) \), then the previous remark leads to solve
\[ \partial_t (u(t, x) - Z(t, x)) = \frac{1}{2} \partial_{xx}^2 (u(t, x) - Z(t, x)) + b(u(t, x) - Z(t, x) + Z(t, x)) \]

a random PDE with \( C^{1/4} \) regularity in translation.
To solve
\[ \partial_t u(t, x) = \frac{1}{2} \partial_{xx}^2 u(t, x) + b(u(t, x)) + \dot{W}(t, x). \] (\star)

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\[ \partial_t Y(t, x) = \frac{1}{2} \partial_{xx}^2 Y(t, x) + b(Y(t, x) + Z(t, x)) \]
a random PDE with \( C^{\frac{1}{4}} \) regularity in translation.
SPDE with singular drift towards SDE with fractional noise

To solve
\[ \partial_t u(t, x) = \frac{1}{2} \partial_{xx}^2 u(t, x) + b(u(t, x)) + \dot{W}(t, x). \]  

Denote \( Y(t, x) = u(t, x) - Z(t, x) \), then the previous remark leads to solve
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a random PDE with \( C_{\frac{1}{4}^-} \) regularity in translation.

Therefore, one can expect that strong existence and uniqueness for equation (\( \ast \)) would hold under the same conditions on \( b \) as needed in a SDE driven by \( \frac{1}{4} \)-fractional Brownian motion.
SPDE with singular drift towards SDE with fractional noise

To solve
\[ \partial_t u(t, x) = \frac{1}{2} \partial_{xx}^2 u(t, x) + b(u(t, x)) + \dot{W}(t, x). \] \hfill (⋆)

Denote \( Y(t, x) = u(t, x) - Z(t, x) \), then the previous remark leads to solve
\[ \partial_t Y(t, x) = \frac{1}{2} \partial_{xx}^2 Y(t, x) + b(Y(t, x) + Z(t, x)) \]

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Therefore, one can expect that strong existence and uniqueness for equation (⋆) would hold under the same conditions on \( b \) as needed in a SDE driven by \( \frac{1}{4} \)-fractional Brownian motion.

That is \( b \in C^\alpha_\gamma / \mathcal{B}_p^{\gamma} \), where \( \alpha = \gamma - 1/p > -1 \).

Note that the Dirac delta function lies in \( \mathcal{B}_p^{-1+1/p} \) which is the critical case.
Figure: Before touching
Figure: Breaks physical limit
Singular model of equations
Sewing approach and Tamed Euler scheme
Tamed Euler scheme for SPDE
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Figure: Penalization steps
Figure: Reflection
Figure: Leaves physical limit
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3. Tamed Euler scheme for SPDE
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Elements of proof: The basic ingredient is the **stochastic sewing lemma**.

**Lemma ([Lê’20])**

Let \( m \in [2, \infty) \). Let \( A : \Delta_{0,1} \rightarrow L^m(\Omega) \) s.t. \( A_{s,t} \) is \( \mathcal{F}_t \)-measurable. Assume

\[ \exists \Gamma_1, \Gamma_2 \geq 0, \text{ and } \varepsilon_1, \varepsilon_2 > 0 \text{ s.t. } \forall (s, t) \in \Delta_{0,1} \text{ and } u := \frac{s + t}{2}, \]

\[
\|E^s[\delta A_{s,u,t}]\|_{L^m} \leq \Gamma_1 (t - s)^{1 + \varepsilon_1},
\]

\[
\|\delta A_{s,u,t}\|_{L^m} \leq \Gamma_2 (t - s)^{1 + \varepsilon_2}.
\]

Then \( \exists (A_t)_{t \in [0,1]} \) s.t. \( \forall t \in [0, 1] \) and any sequence of partitions \( \Pi_k = \{t^k_i\}_{i=0}^{N_k} \) of \([0, t]\) with mesh size going to zero,

\[
A_t = \lim_{k \to \infty} \sum_{i=0}^{N_k} A_{t^k_i, t^k_{i+1}} \text{ in proba.}
\]

Moreover, \( \exists C \) s.t. \( \forall (s, t) \in \Delta_{0,1} \),

\[
\|A_t - A_s - A_{s,t}\|_{L^m} \leq C \Gamma_1 (t - s)^{1 + \varepsilon_1} + C \Gamma_2 (t - s)^{1 + \varepsilon_2},
\]

\[
\|E^s[A_t - A_s - A_{s,t}]\|_{L^m} \leq C \Gamma_1 (t - s)^{1 + \varepsilon_1}.
\]
Elements of proof

It leads to key estimates for “smooth” $f$:

- $(\psi_t)$ is $\mathbb{F}$-adapted, $m \in [2, \infty)$, $p \in [m, \infty]$ and $\gamma < 0$ s.t. $(\gamma - d/p) > -1/(2H)$. Let $\alpha \in (0, 1)$ s.t. $H(\gamma - d/p - 1) + \alpha > 0$,

$$\left\| \int_s^t f(B_r + \psi_r) \, dr \right\|_{L^m(\Omega)} \leq C \left\| f \right\|_{\mathcal{B}^\gamma_p} (t - s)^{1 + H(\gamma - d/p)}$$

$$+ C \left\| f \right\|_{\mathcal{B}^\gamma_p} [\psi]_{\mathcal{C}^{\alpha}_{[s,t]}} L^m (t - s)^{1 + H(\gamma - d/p - 1) + \alpha}.$$  

$\rightarrow$ leads to existence via a tightness-stability argument.
Elements of proof

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- $(\psi_t)$ is $\mathbb{F}$-adapted, $m \in [2, \infty)$, $p \in [m, \infty]$ and $\gamma < 0$ s.t. $(\gamma - d/p) > -1/(2H)$. Let $\alpha \in (0, 1)$ s.t. $H(\gamma - d/p - 1) + \alpha > 0$,

$$
\left\| \int_s^t f(B_r + \psi_r) \, dr \right\|_{L^m(\Omega)} \leq C \| f \|_{B_p^\gamma} (t - s)^{1 + H(\gamma - d/p)} \\
+ C \| f \|_{B_p^\gamma} \| \psi \|_{L^m([s, t])} (t - s)^{1 + H(\gamma - d/p - 1) + \alpha}.
$$

(1)

$\rightarrow$ leads to existence via a tightness-stability argument.

- $m \in [2, \infty)$ and $p \in [m, \infty]$ and $0 > \gamma > 1 - 1/(2H)$. For any $\mathcal{F}_s$-measurable $\kappa_1, \kappa_2 \in L^m(\Omega)$,

$$
\left\| \int_s^t f(B_r + \kappa_1) - f(B_r + \kappa_2) \, dr \right\|_{L^m} \\
\leq C \| f \|_{B_p^\gamma} \| \kappa_1 - \kappa_2 \|_{L^m(t - s)^{1 + H(\gamma - d/p - 1)}},
$$

(2)

(for uniqueness).
Why the condition $\gamma - \frac{d}{p} > \frac{1}{2} - \frac{1}{2H}$ in the theorem?

Consider $(b^n)$ in $C^\infty \cap B_p^\gamma$ that approximates $b$. Let $X^n$ be the solution with drift $b^n$. 

Elements of proof II
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Consider $(b^n)$ in $C^\infty \cap B_p^{\gamma}$ that approximates $b$. Let $X^n$ be the solution with drift $b^n$.

1) We look for a priori estimate on the Hölder regularity of $X^n - B$. If uniform in $n$, we get tightness.
Elements of proof II

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1) We look for a priori estimate on the Hölder regularity of $X^n - B$. If uniform in $n$, we get tightness.

2) In (1), replace $f$ by $b^n$, $X$ by $X^n$ and $K^n_t := \int_0^t b^n(X_s)ds \equiv \psi_t$. Then for any $\alpha$ such that $H(\gamma - d/p - 1) + \alpha > 0$,

$$
\|K^n_t - K^n_s\|_{L^m(\Omega)} \leq C \|b^n\|_{B_p^\gamma} (t - s)^{1 + H(\gamma - d/p)}
+ C \|b^n\|_{B_p^\gamma} [K^n]_{C^\alpha_{[s,t]}} L^m(t - s)^{1 + H(\gamma - d/p - 1) + \alpha}.
$$
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$$
\|K^n_t - K^n_s\|_{L^m(\Omega)} \leq C\|b^n\|_{B_p^{\gamma}}(t - s)^{1 + H(\gamma - d/p)} + C\|b^n\|_{B_p^{\gamma}}[K^n]_{C_{[s,t]}^{\alpha}}L^m(t - s)^{1 + H(\gamma - d/p - 1) + \alpha}.
$$

3) Choosing $\alpha = 1 + H(\gamma - d/p)$ above with $H(\gamma - d/p - 1) + \alpha > 0$ requires $\gamma - \frac{d}{p} > \frac{1}{2} - \frac{1}{2H}$. Then for $t - s$ small enough,

$$
[K^n]_{C_{[s,t]}^{\alpha}}L^m \leq C\|b^n\|_{B_p^{\gamma}} + \frac{1}{2}[K^n]_{C_{[s,t]}^{\alpha}}L^m.
$$

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New way of proof
**Elements of proof II**

**Why the condition** \( \gamma - \frac{d}{p} > \frac{1}{2} - \frac{1}{2H} \) **in the theorem?**

Consider \((b^n)\) in \(C^\infty \cap B_p^\gamma\) that approximates \(b\). Let \(X^n\) be the solution with drift \(b^n\).

1. We look for a priori estimate on the Hölder regularity of \(X^n - B\). If uniform in \(n\), we get tightness.

2. In (1), replace \(f\) by \(b^n\), \(X\) by \(X^n\) and \(K_t^n := \int_0^t b^n(X_s)ds \equiv \psi_t\). Then for any \(\alpha\) such that \(H(\gamma - d/p - 1) + \alpha > 0\),

\[
\|K_t^n - K_s^n\|_{L^m(\Omega)} \leq C \|b^n\|_{B_p^\gamma} (t - s)^{1 + H(\gamma - d/p)}
\]

\[
+ C \|b^n\|_{B_p^\gamma} [K^n]_{\alpha}^{C_{\alpha,t}} L^m (t - s)^{1 + H(\gamma - d/p - 1) + \alpha}.
\]

3. Choosing \(\alpha = 1 + H(\gamma - d/p)\) above with \(H(\gamma - d/p - 1) + \alpha > 0\) requires \(\gamma - \frac{d}{p} > \frac{1}{2} - \frac{1}{2H}\). Then for \(t - s\) small enough,

\[
[K^n]_{\alpha}^{C_{\alpha,t}} L^m \leq C \|b^n\|_{B_p^\gamma} + \frac{1}{2} [K^n]_{\alpha}^{C_{\alpha,t}} L^m.
\]

4. Extend to any \(s \leq t\) and get tightness.
1. Singular model of equations
   - Numerical simulations
   - SDE Weak solutions

2. Sewing approach and Tamed Euler scheme
   - Sewing approach
   - Tamed Euler scheme
   - Strong existence and uniqueness of the SDE

3. Tamed Euler scheme for SPDE
   - Finite differences
   - Controlling the error
Let $h > 0$ and $(b^n)$ that approximates $b$ in $\mathcal{B}_p^\gamma(\mathbb{R}^d)$. Consider the following tamed Euler scheme:

$$X^{h,n}_t = X_0 + \int_0^t b^n(X^{h,n}_{r_h})dr + B_t,$$

where $r_h = h\lfloor \frac{r}{h} \rfloor$. 


"Subcritical" case

**Theorem ([G.-Haress-Richard.'22])**

Let $H < \frac{1}{2}$, $m \geq 2$ and $p \in [m, \infty]$. Let $b \in \mathcal{B}_p^\gamma$ and assume

$$0 > \gamma - \frac{d}{p} > 1 - \frac{1}{2H}.$$ 

Let $X$ denote a weak solution such that $X - B \in C_{[0,T]}^{\frac{1}{2}+H}(L^m)$. Let $\varepsilon \in (0, \frac{1}{2})$. Then $\forall h \in (0, 1)$ and $\forall n \in \mathbb{N}$,

$$[X_t - X_t^{h,n}]_{C^{\frac{1}{2}+H}} \leq C \left( \|b^n - b\|_{\mathcal{B}_p^{\gamma-1}} + \|b^n\|_{\infty} h^{\frac{1}{2} - \varepsilon} \right. \left. + \|b^n\|_{\infty} \|b^n\|_{C^{1}h^{1-\varepsilon}} \right).$$
Denote by \( G_t \) the Gaussian semigroup with variance \( t \). Then

\[
\| G_{\frac{1}{n}} b - b \|_{B_p^{\gamma-1}} \lesssim n^{-\frac{1}{2}} \| b \|_{B_p^{\gamma}}, \\
\| G_{\frac{1}{n}} b \|_{\infty} \lesssim n^{\frac{1}{2}(-\gamma + \frac{d}{p})} \| b \|_{B_p^{\gamma}}, \\
\| G_{\frac{1}{n}} b \|_{C^1} \lesssim n^{\frac{1}{2}(1 - \gamma + \frac{d}{p})} \| b \|_{B_p^{\gamma}}.
\]

**Corollary**

Let \( h \in (0, \frac{1}{2}) \), \( n_h = \lfloor h^{\frac{1}{1 - \gamma + \frac{d}{p}}} \rfloor \) and \( b^{n_h} = G_{\frac{1}{n_h}} b \). Then

\[
[X_t - X_t^{h,n_h}]_{C^{\frac{1}{2}}} \lesssim C h^{2(1 - \gamma + \frac{d}{p}) - \varepsilon}.
\]
“Critical” case

**Theorem ([G.-Haress-Richard.’22])**

Let $H < \frac{1}{2}$, $m \geq 2$ and $p \in [m, \infty]$. Let $b \in B_p^\gamma$ and assume

$$\gamma - \frac{d}{p} = 1 - \frac{1}{2H} \text{ and } p < +\infty.$$ 

Let

$$\epsilon(h, n) = \|b - b^n\|_{B_p^{\gamma-1}} (1 + |\log(\|b - b^n\|_{B_p^{\gamma-1}})|)$$

$$+ \|b^n\|_{\infty} h^{\frac{1}{2} - \epsilon} + \|b^n\|_{C_1} \|b^n\|_{\infty} h^{1-\epsilon}.$$ 

Then $\exists C, \rho > 0$ such that for all $h \in (0, 1)$ and $n \in \mathbb{N}$,

$$[X_t - X_t^{h,n}]_{C^\frac{1}{2} - L^m} \leq C\epsilon(h, n)^\rho.$$
Rates

The orders of convergence obtained here compared to regular $b$ from Butkovsky et al. (2021a); Dareiotis at al. (2021); De Angelis et Al. (2019) are:

<table>
<thead>
<tr>
<th>Drift</th>
<th>Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha &gt; 0$</td>
<td>$\left(\frac{1}{2} + H\alpha\right) \wedge 1$</td>
</tr>
<tr>
<td>$\gamma - \frac{d}{p} &gt; 0$</td>
<td>$\left(\frac{1}{2} + H \left(\gamma - \frac{d}{p}\right)\right) \wedge 1 - \varepsilon$</td>
</tr>
<tr>
<td>$\alpha = 0 \sim$ Bounded</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>$\gamma - \frac{d}{p} = 0$</td>
<td>$\frac{1}{2} - \varepsilon$</td>
</tr>
<tr>
<td>$\gamma - \frac{d}{p} \in \left(1 - \frac{1}{2H}, 0\right)$</td>
<td>$\frac{1}{2(1 - \gamma + \frac{d}{p}) - \varepsilon}$</td>
</tr>
<tr>
<td>$\gamma - \frac{d}{p} = 1 - \frac{1}{2H}$ and $p &lt; \infty$</td>
<td>$\rho = He^{-M} - \varepsilon$</td>
</tr>
</tbody>
</table>
Define

\[ K^n_t := \int_0^t b^n(X_r)dr \quad \text{and} \quad K^{h,n}_t := \int_0^t b^n(X^{h,n}_r)dr. \]

Decompose the error as follows:

\[ X_t - X^{h,n}_t - (X_s - X^{h,n}_s) = K_t - K_s - (K^n_t - K^n_s) \quad (3) \]

\[ + \int_s^t (b^n(K_r + B_r) - b^n(K^{h,n}_r + B_r))dr \quad (4) \]

\[ + \int_s^t (b^n(K^{h,n}_r + B_r) - b^n(K^{h,n}_r + B_r))dr. \quad (5) \]

Denote by \( E^{1,h,n}_{s,t} \) the term in (4), \( E^{2,h,n}_{s,t} \) the term in (5), and \( E^{h,n}_{s,t} = E^{1,h,n}_{s,t} + E^{2,h,n}_{s,t} \).
Elements of proof II

The terms (3) and (4) are controlled "classically" by stochastic sewing:

\[
[K - K^n] \frac{1}{C_{[s, t]}^\frac{1}{2}} L^m \leq C \| b - b^n \|_{B_p^{\gamma - 1}}
\]

and

\[
\| E_{s, t}^{1, h, n} \|_{L^m} \leq C ([E^{h, n}] \frac{1}{C_{[s, t]}^\frac{1}{2}} L^m + \| b - b^n \|_{B_p^{\gamma - 1}}) (t - s)^{1 + H(\gamma - \frac{d}{p} - 1)}.
\]
Elements of proof II

The terms (3) and (4) are controlled “classically” by stochastic sewing:

\[
[K - K^n] C_{[s, t]}^{1/2} L^m \leq C \| b - b^n \|_{B^{\gamma - 1}}
\]

and

\[
\| E^{1, h, n}_{s, t} \|_{L^m} \leq C (\| E^{h, n}_{s, t} \|_{L^m} + \| b - b^n \|_{B^{\gamma - 1}})(t - s)^{1 + H(\gamma - \frac{d}{p} - 1)}.
\]

The last term \( E^{2, h, n}_{s, t} \) can be controlled using Girsanov, but this leads to an exponential dependence in \( \| b^n \|_\infty \). Instead with use again the SSL to get

\[
\| E^{2, h, n}_{s, t} \|_{L^m} \leq C \left( (\| b^n \|_\infty + 1) h^{1/2} (t - s)^{1 + H(\gamma - \frac{d}{p} - 1)} + \| b^n \|_\infty h^{1/2 - \varepsilon} | t - s |^{1/2 + \varepsilon} + \| b^n \|_{C^1} \| b^n \|_\infty h^{1 - \varepsilon} (t - s)^{1 + \varepsilon} \right).
\]
1 Singular model of equations
- Numerical simulations
- SDE Weak solutions

2 Sewing approach and Tamed Euler scheme
- Sewing approach
- Tamed Euler scheme
- Strong existence and uniqueness of the SDE

3 Tamed Euler scheme for SPDE
- Finite differences
- Controlling the error
Consequences

Definition

- **Pathwise uniqueness** holds if for any solutions \((X, B)\) and \((Y, B)\) defined on the same filtered probability space with the same \(B\) and same initial condition \(X_0\), \(X\) and \(Y\) are indistinguishable.

- A weak solution \((X, B)\) such that \(X\) is \(\mathbb{F}^B\)-adapted is called a **strong solution**.
Strong existence and uniqueness

As a consequence of the convergence of the Euler scheme,

**Theorem ([G.-Haress-Richard.’22])**

Let $H < \frac{1}{2}$, $\gamma \in \mathbb{R}$, $p \in [1, \infty]$ and $b \in \mathcal{B}_p^\gamma$. Assume that

$$0 > \gamma - \frac{d}{p} \geq 1 - \frac{1}{2H} \quad \text{and} \quad \gamma > 1 - \frac{1}{2H}.$$

- There exists a strong solution $X$ to (*) such that $[X - B]_{C^{1/2+H}_{[0,1],\infty}} < \infty$ for any $m \geq 2$.
- Pathwise uniqueness holds in the class of solutions such that $[X - B]_{C^{1/2+H}_{[0,1],\infty}} < \infty$. 

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New way of proof
As a consequence of the convergence of the Euler scheme,

**Theorem ([G.-Haress-Richard.'22])**

Let $H < \frac{1}{2}$, $\gamma \in \mathbb{R}$, $p \in [1, \infty]$ and $b \in \mathcal{B}_{\gamma}^{p}$. Assume that

$$0 > \gamma - \frac{d}{p} \geq 1 - \frac{1}{2H} \quad \text{and} \quad \gamma > 1 - \frac{1}{2H}.$$  

- There exists a strong solution $X$ to (*) such that $[X - B]_{C^{1/2+H}_{[0,1]}}^{1/2+H} \leq \infty$ for any $m \geq 2$.
- Pathwise uniqueness holds in the class of solutions such that $[X - B]_{C^{1/2+H}_{[0,1]}}^{1/2+H} \leq \infty$.
- If $\gamma - \frac{d}{p} > 1 - \frac{1}{2H}$, for all $\eta \in (0, 1)$, pathwise uniqueness holds in the class of solutions such that $[X - B]_{C^{H(1 - \gamma + d/p) + \eta}_{[0,1]}}^{H(1 - \gamma + d/p) + \eta} \leq \infty$. 
1. Singular model of equations
   - Numerical simulations
   - SDE Weak solutions

2. Sewing approach and Tamed Euler scheme
   - Sewing approach
   - Tamed Euler scheme
   - Strong existence and uniqueness of the SDE

3. Tamed Euler scheme for SPDE
   - Finite differences
   - Controlling the error
1. Singular model of equations
   - Numerical simulations
   - SDE Weak solutions

2. Sewing approach and Tamed Euler scheme
   - Sewing approach
   - Tamed Euler scheme
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3. Tamed Euler scheme for SPDE
   - Finite differences
   - Controlling the error
Stochastic PDEs

Consider

$$\partial_t u(t, x) = \frac{1}{2} \partial_{xx}^2 u(t, x) + b(u(t, x)) + \xi$$

where $\xi$ is a space-time white noise and $b$ is a distribution in some Besov space, with bounded measurable initial data $\psi_0$.

In [Athreya et al.’20], it is proven that there exists a weak solution with certain regularity. The goal is to approximate it.
Consider
\[ \partial_t u(t, x) = \frac{1}{2} \partial_{xx} u(t, x) + b(u(t, x)) + \xi \]  

where \( \xi \) is a space-time white noise and \( b \) is a distribution in some Besov space, with bounded measurable initial data \( \psi_0 \).

In [Athreya et al.'20], it is proven that there exists a weak solution with certain regularity. The goal is to approximate it.

Consider a sequence \( b^k \) of smooth function which approximates the distribution \( b \) in some sense. We say that a sequence of smooth bounded functions \( (b^k) \) converges to \( b \) in \( B_p^{\gamma^-} \) as \( k \) goes to infinity if

\[
\begin{cases}
\sup_{k \in \mathbb{N}} \| b^k \|_{B_p^{\gamma}} < \| b \|_{B_p^{\gamma}} < \infty \\
\lim_{k \to \infty} \| b^k - b \|_{B_p^{\gamma'}} = 0 \quad \forall \gamma' < \gamma.
\end{cases}
\]
Definition

A couple \(((u_t(x))_{t \in [0,1]}, \xi)\) is a weak solution on some filtered space \((\Omega, \mathbb{P}, \mathcal{F})\) if there exists a process \(K : [0, 1] \times [0, 1] \times \Omega \to \mathbb{R}\) such that

1. \(\xi\) is an \(\mathcal{F}\)-space time white noise.
2. \(u\) is adapted to \(\mathcal{F}\).
3. \(u_t(x) = P_t \psi_0(x) + K_t(x) + O_t(x)\) a.s where \(x \in [0, 1], t \in [0, 1]\).
4. For any sequence \((b^k)_{k \in \mathbb{N}}\) in \(\mathcal{C}^b\) converging to \(b\) in \(B_p^{\gamma-}\), we have

\[
\sup_{t \in [0, 1]} \left| \int_0^t \int_0^1 p_{t-r}(x, y)b^k(u_r(y))dydr - K_t(x) \right| \xrightarrow{k \to \infty} 0.
\]

5. Almost surely, the function \(u\) is continuous on \([0, 1] \times [0, 1]\).

If the couple is clear from the context, we simply say that \(u\) is a weak solution.
The mild form associated to the SPDE is

\[ u_t(x) = P_t \psi_0(x) + \int_0^t \int_0^1 p_{t-r}(x, y) b(u_r(y)) dy dr + O_t(x) . \]

Notice that the mild form is also not well-posed when b is a genuine distribution. To define a solution, we first specify in which spaces we take \( b \), and then define how we approximate \( b \) via a smooth sequence.
We encounter different heat kernels: the continuum Gaussian on $\mathbb{R}$,
\[ g_t(x) = \frac{1}{\sqrt{2\pi t}} \exp \left( -\frac{x^2}{2t} \right), \]
the continuum heat kernel on $[0, 1]$ associated with the boundary conditions
\[ p_t(x, y) = p_t(x-y) = \sum_{k \in \mathbb{Z}} \frac{1}{\sqrt{4\pi t}} \exp \left( -\frac{(x-y+k)^2}{4t} \right) = \sum_{k \in \mathbb{Z}} e^{-4\pi^2 k^2 t} e^{i2\pi k(x-y)}. \]

We denote by $G_\cdot$ and $P_\cdot$ the respective convolutions with the $g_\cdot$ and $p_\cdot$.

That is, for any bounded measurable function $f$, we write
\[ G_t f(x) = \int_0^1 g_t(x-y) f(y) dy, \quad \text{and} \quad P_t f(x) = \int_0^1 p_t(x-y) dy. \]

We also denote the convolution operator by $(f \ast g)(x) = \int f(x-y)g(y) dy$. 

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New way of proof
We denote by $Q(t)$ the variance of the Ornstein-Uhlenbeck process

$$O_t(x) = \int_0^t \int_0^1 p_{t-r}(x, y) \xi(dy, dr).$$

In fact, for all $x \in [0, 1]$

$$\mathbb{E}(O_t(x)^2) = \int_0^t \int_0^1 p_t(x-y)^2 dydr = \int_0^t \int_0^1 p_{2t}(y)dydr =: Q(t).$$

Moreover, writing $Z_{s,t}(x) = O_t(x) - P_{t-s}O_s(x)$ for $t \geq s$, we have that the random variable $Z_{s,t}(x)$ is independent of $\mathcal{F}_s$ and

$$\mathbb{E}\left(Z_{s,t}(x)^2\right) = Q(t-s).$$

The following equality will be used a lot. For any continuous function $h$ and $\mathcal{F}_s$-measurable random variable $Y$ one has the almost sure equality

$$\mathbb{E}^s h(O_t(x) + Y) = G_{Q(t-s)}h(P_{t-s}O_s(x) + Y).$$
We study a tamed Euler finite-difference.
Finite differences

We study a tamed Euler finite-difference.

Fix $k \in \mathbb{N}$, such that $b^k$ is close to $b$. 
Finite differences

We study a tamed Euler finite-difference.

Fix $k \in \mathbb{N}$, such that $b^k$ is close to $b$.

Let $h \in (0, 1)$ be a time step of the form $h = c(2n)^{-2}$ for some $n \in \mathbb{N}$, where $c$ is a constant satisfying the CFL condition $c > \frac{1}{2}$.
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We introduce the following space and time grids.

$$\Pi_n = \{0, (2n)^{-1}, \ldots, (2n - 1)(2n)^{-1}\}, \quad \Lambda_h = \{0, h, 2h, \ldots\}.$$
Finite differences

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We introduce the following space and time grids.

$$
\Pi_n = \left\{ 0, (2n)^{-1}, \ldots, (2n-1)(2n)^{-1} \right\}, \quad \Lambda_h = \{ 0, h, 2h, \ldots \}.
$$

We can now define the numerical scheme for $x \in \Pi_n$ and $t \in \Lambda_h$ as

$$
\begin{cases}
    u^{h,k}_{t+h}(x) = u^{h,k}_t(x) + h\Delta_n u^{h,k}_t(x) + h b^k \left( u^{h,k}_t(x) \right) + h \eta_{h,n}(t, x) \\
    u^{h,k}_0(x) = \psi_0(x),
\end{cases}
$$
Finite differences

\( \Delta_n \) is the discrete Laplacian

\[
\Delta_n f(x) = (2n)^2 \left( f \left( x + (2n)^{-1} \right) - 2f(x) + f \left( x - (2n)^{-1} \right) \right),
\]

and discrete noise \( \eta_{h,n} \) is given by

\[
\eta_h(t, x) = (2n) h^{-1} \xi (\left[ t, t + h \right] \times \left[ x, x + (2n)^{-1} \right]).
\]
The goal is the rate of convergence

Rate of convergence

For the right choice of $b^k$ and $k$, we prove the following rate of convergence for the approximation $u^{h,k}$:

$$\sup_{t \in \Lambda_h, x \in \Pi_n} \| u_t - u_t^{h,k} \|_{L^m(\Omega)} \leq C \left( n^{-\frac{1}{2(1-\gamma+\frac{1}{p})}} \right).$$
Consider the functions $e_j(x) = e^{i2\pi jx}$ for $j \in \mathbb{Z}$. They are eigenfunctions of $\Delta$ with eigenvalues $\lambda_j = -4\pi^2 j^2$. It is well known that $(e_j)_{j \in \mathbb{Z}}$ forms an orthonormal basis of $L^2([0, 1], \mathbb{C})$.

The eigenvalues of the discrete Laplacian $\Delta_h$ are

$$\lambda_j^n = -16n^2 \sin^2 \left( \frac{j\pi}{2n} \right) \text{ for } j \in \mathbb{Z}.$$
We know that for $-n \leq j, l \leq n - 1$,

$$\Delta_h e_j(x) = \lambda^h_j e_j(x),$$

and

$$\frac{1}{2n} \sum_{x \in \Pi_n} e_j(x) e_{l}(x) = 1_{j=l}.$$

As a consequence, $e_i$ for $i \in [-n, n - 1]$, as functions on $\Pi_n$ form a basis of $L^2(\Pi_n; \mathbb{C})$.

It will be convenient to use the piecewise linear extension of the restriction of $e_j$ to $\Pi_n$: for $-n \leq j \leq n - 1$, for $x \in \Pi_n$, and $x' \in [x, x + (2n)^{-1}]$, set

$$e_j(x') = e_j(x) + (2n) (x' - x) \left( e_j(x + (2n)^{-1}) - e_j(x) \right).$$
It remains to encode the temporal discretisation.

Naturally, on the temporal gridpoints $t = kh$ a factor $(1 + h\lambda_j^n)^k$ appears.

Between the gridpoints, we again interpolate linearly. More precisely, for $j = -n, \ldots, n - 1$, for $t \in \Lambda_h$, and $t' \in [t, t + h]$, set

$$\mu_j^h (t') = (1 + h\lambda_j^n)^{th^{-1}} + h^{-1} (t' - t) \left( (1 + h\lambda_j^n)^{(t+h)h^{-1}} - (1 + h\lambda_j^n)^{th^{-1}} \right).$$
Discrete mild form

We can now define the discrete heat kernel and discrete convolution. Our goal is to write the numerical scheme in a mild form similar to classical mild form for SPDEs.

For $t \in [0, 1]$ and $x \in [0, 1]$, denote by $t_h$ and $x_n$ the leftmost gridpoint from $t$ in $\Lambda_h$ and from $x$ in $\Pi_n$ respectively.

The mild form associated to the scheme is

$$u_t^{h,k}(x) = P_t^n \psi_0(x) + \int_0^t P^n_{(t-s)_h} b^k (u_{s_h}^{h,k})(x) ds + \int_0^t \int_0^1 p^n_{(t-s)_h}(x, y) \xi(dy, ds)$$

where $P^n$ and $p^n$ are respectively the discrete analogues of $P$ and $p$. 

Discrete mild form

They are defined for all \( t \in [0, 1] \) and \( x \in [0, 1] \)

\[
P_t^n f(x) := p_t^n \ast_n f \quad \text{and} \quad p_t^n(x, y) = \sum_{j=-n}^{n-1} \mu_j^n(t)e^n_j(x)e^n_j(y_n).
\]

Moreover, \( \ast_n \) denotes the discrete convolution

\[
g \ast_n f(x) := \int_0^1 g(x - y)f(y_n)dy.
\]
Ornstein-Uhlenbeck process

**Definition**

We define the discrete Ornstein-Uhlenbeck process as

\[ O^n_t(x) := \int_0^t \int_0^1 p_{(t-r)_h}^h(x, y) \xi(dy, dr), \]

Analogously to \( O_t \), for \( s \leq t \), we define \( \hat{O}^h_{s,t} \) and \( Z^h_{s,t} \) by

\[ O^n_t(x) = \int_0^s \int_0^1 p_{(t-r)_h}^n(x, y) \xi(dy, dr) + \int_s^t \int_0^1 p_{(t-r)_n}^n(x, y) \xi(dy, dr) \]

\[ =: \hat{O}^n_{s,t} + Z^n_{s,t}. \]
Notice that $Z_{s,t}^n(x)$ is a random variable independent of $\mathcal{F}_s$ and has variance

$$
\mathbb{E}\left(Z_{s,t}^n(x)^2\right) = \int_0^{t-s} \int_0^1 |p_{rn}^n(x,y)|^2 \, dy \, dr
$$

$$
= \int_0^{t-s} \sum_{j=-n}^{n-1} |1 + h\lambda_j^n|^{2rh^{n-1}} \, dr =: Q^n(t-s).
$$

Also similarly to $O_t$, we have that for any continuous function $h$ and $\mathcal{F}_s$-measurable random variable $Y$

$$
\mathbb{E}^s h\left(O^n_t(x) + Y\right) = G_{Q^n(t-s)} h\left(\widehat{O}^n_{s,t}(x) + Y\right).
$$
1. **Singular model of equations**
   - Numerical simulations
   - SDE Weak solutions

2. **Sewing approach and Tamed Euler scheme**
   - Sewing approach
   - Tamed Euler scheme
   - Strong existence and uniqueness of the SDE

3. **Tamed Euler scheme for SPDE**
   - Finite differences
   - Controlling the error
We are interested in controlling the error $\mathcal{E}^{h,k}_{s,t}$ between the solution $u$ and the numerical scheme $u^{h,k}$, defined for $(s,t) \in \Delta_{0,1}$ and $x \in [0,1]$ by

$$\mathcal{E}^{h,k}_{s,t}(x) = u_t(x) - P_{t-s}u_s(x) - \int_s^t P^n_{(t-s)_h} b^k(u^{h,k}_{r_h})(x) dr.$$
Elements of Proof I

For all \((s, t) \in \Delta_{0,1}\) and \(x \in [0, 1]\), we write

\[
\mathcal{E}_{s,t}^{h,k}(x) = v_t(x) - P_{t-s}v_s - \int_{s}^{t} P_{(t-r)} b^k (v_{r}^{h,k} + P^n_r \psi_0 + O^n_r)(x)\,ds
\]

\[
= v_t(x) - v_t^k(x) - P_{t-s}v_s(x) + P_{t-s}v_s^k(x)
\]

\[
+ \int_{s}^{t} \left( P_{(t-r)} b^k (v_r + P_r \psi_0 + O_r)(x) \right)\,ds
\]

\[
- \int_{s}^{t} P_{(t-r)} b^k (v_{r}^{h,k} + P^n_r \psi_0 + O^n_r(x))(x)\,ds
\]

\[
:= V_t^k - P_{t-s}V_s^k + \mathcal{E}_{s,t}^{1,h,k} + \mathcal{E}_{s,t}^{2,h,k} + \mathcal{E}_{s,t}^{3,h,k},
\]

\[
:= \epsilon(h, k) + \mathcal{E}_{s,t}^{1,h,k}
\]
Elements of Proof II

\[ \mathcal{E}^{1,h,k}_{s,t} = \int_0^t \int_0^1 p_{t-r}(x, y) \left( b^k(v_r(y) + P_r \psi_0(y) + O_r(y)) \right. \]
\[ \left. - b^k(v^h_{r,k}(y) + P^n_r \psi_0(y) + O^n_r(y)) \right) dy dr \]

\[ \mathcal{E}^{2,h,k}_{s,t} = \int_0^t \int_0^1 p_{t-r}(x, y) \left( b^k(v^h_{r,k}(y) + P^n_r \psi_0(y) + O^n_r(y)) \right. \]
\[ \left. - b^k(v_{r,k}(y_n) + P^n_{r_n} \psi_0(y_n) + O^n_{r_n}(y_n)) \right) dy dr \]

\[ \mathcal{E}^{3,h,k}_{s,t} = \int_0^t \int_0^1 (p_{t-r} - p^n_{(t-r)_h})(x, y) b^k(v^h_{r,k}(y_n) + P^n_{r_n} \psi_0(y_n) + O^n_{r_n}(y_n)) dy dr. \]

will be controlled on small intervals.
Elements of Proof III

On all intervals $[S, T]$, we have

$$[V^k]\frac{1}{C_{[S,T]}^\frac{1}{2}}L^m \leq C\| b - b^k \|_{B^{\gamma-1}}.$$

$$[E^{2,h,k}]\frac{1}{C_{[S,T]}^\frac{1}{2}}L^m + [E^{3,h,k}]\frac{1}{C_{[S,T]}^\frac{1}{2}}L^m \leq C \left( (1 + \| b^k \|_{\infty})n^{-\frac{1}{2}+\varepsilon} + (1 + \| b^k \|_{\infty})(1 + \| b^k \|_{C^1})n^{-1+\varepsilon} \right).$$
Elements of Proof III

On all intervals \([S, T]\), we have

\[
[V^k]_{C^{\frac{1}{2}}_{[S, T]}L^m} \leq C\|b - b^k\|_{B^{\gamma -1}_p}.
\]

\[
[\mathcal{E}^{2, h, k}]_{C^{\frac{1}{2}}_{[S, T]}L^m} + [\mathcal{E}^{3, h, k}]_{C^{\frac{1}{2}}_{[S, T]}L^m} \\
\leq C \left( (1 + \|b^k\|_{\infty})n^{-\frac{1}{2} + \varepsilon} + (1 + \|b^k\|_{\infty})(1 + \|b^k\|_{C^1})n^{-1 + \varepsilon} \right).
\]

which reads

\[
\varepsilon(h, k) \leq C \left( \|b - b^k\|_{B^{\gamma -1}_p} + (1 + \|b^k\|_{\infty})n^{-\frac{1}{2} + \varepsilon} + (1 + \|b^k\|_{\infty})(1 + \|b^k\|_{C^1})n^{-1 + \varepsilon} \right).
\]
Elements of Proof III

On all intervals \([S, T]\), we have

\[
\left[ V^k \right]_{C_{[S, T]}^{\frac{1}{2}}} \leq C \| b - b^k \|_{B^\gamma_1 - 1}.
\]

\[
\left[ \mathcal{E}^{2,h,k} \right]_{C_{[S, T]}^{\frac{1}{2}}} \left[ \mathcal{E}^{3,h,k} \right]_{C_{[S, T]}^{\frac{1}{2}}} \leq C \left( (1 + \| b^k \|_\infty) n^{-\frac{1}{2} + \epsilon} + (1 + \| b^k \|_\infty)(1 + \| b^k \|_{C^1}) n^{-1 + \epsilon} \right).
\]

which reads

\[
\epsilon(h, k) \leq C \left( \| b - b^k \|_{B^{\gamma - 1}} + (1 + \| b^k \|_\infty) n^{-\frac{1}{2} + \epsilon} + (1 + \| b^k \|_\infty)(1 + \| b^k \|_{C^1}) n^{-1 + \epsilon} \right).
\]

The last step is to obtain

\[
\left[ \mathcal{E}^{1,h,k} \right]_{C_{[S, T]}^{\frac{1}{2}}} \leq C \left( \left[ \mathcal{E}^{h,k} \right]_{C_{[S, T]}^{\frac{1}{2}}} + \left[ \mathcal{E}^{h,k} \right]_{C_{[0,S]}^{\frac{1}{2}}} + \epsilon(h, k) \right) (T - S)^{\frac{1}{4}(\gamma + 1 - 1/p)}.
\]
Hence the bound
\[
\left[ \mathcal{E}^{h,k} \right]_{C^{\frac{1}{2}}_{[S,T]} L^m} \leq \epsilon(h,k) + C\left( \left[ \mathcal{E}^{h,k} \right]_{C^{\frac{1}{2}}_{[S,T]} L^m} + \left[ \mathcal{E}^{h,k} \right]_{C^{\frac{1}{2}}_{[0,S]} L^m} \right) (T - S)^{\frac{1}{4}(\gamma+1-1/p)}.
\]

leads to
\[
\left[ \mathcal{E}^{h,k} \right]_{C^{\frac{1}{2}}_{[0,1]} L^m} \leq C \epsilon(h,k),
\]

which is controlled.
Let $\gamma \in \mathbb{R}$, $p \in [1, \infty]$ such that

$$0 > \gamma - \frac{1}{p} \geq -1 \quad \text{and} \quad \gamma > -1. \quad (A1)$$

Let $b \in \mathcal{B}^\gamma_p$, $m \in [2, \infty)$, $\varepsilon \in (0, 1/2)$ and let $\psi_0 \in C^{\frac{1}{2} - \varepsilon}([0, 1], \mathbb{R})$.

Let $u$ be the strong solution to the SPDE with drift $b$.

Let $(b^k)$ be a sequence of smooth functions that converges to $b$ in $\mathcal{B}^\gamma_p$ and $(u^{h,k})_{h \in (0,1), k \in \mathbb{N}}$ be the tamed Euler finite-differences scheme defined on the same probability space and with the same space-time white noise $\xi$ as $u$. 
Theoretical Results

Theorem (G.-Haress-Richard)

(a) Regularity of the tamed Euler scheme: Let \( \eta \in \left( 0, \frac{1}{2} \right) \), \( \mathcal{D} \) be a subset of \([0, 1] \times \mathbb{N}\) and assume that

\[
\sup_{(h,k) \in \mathcal{D}} \| b^k \|_{\infty} h^{\frac{1}{4} - \eta} < \infty \quad \text{and} \quad \sup_{(h,k) \in \mathcal{D}} \| b^k \|_{C^1} h^{\frac{1}{2}} < \infty,
\]

then

\[
\sup_{(h,k) \in \mathcal{D}} \{ u^{h,k} - O \} \leq C_{\frac{1}{2} + \eta} L^m, \infty < \infty.
\]
Theoretical Results

Theorem (G.-Haress-Richard)

Assume that $[u - O]_{C^{3/4}_{[0,1]} L^m, \infty} < \infty$.

(b) The sub-critical case: If $-1 < \gamma - d/p < 0$, then there exists a constant $C$ that depends on $m, p, \gamma, \varepsilon, \|b\|_{B^\gamma}, \|\psi_0\|_{C^{1/2 - \varepsilon}}$ such that for all $h \in (0, 1)$ and $k \in \mathbb{N}$, the following bound holds:

$$[\mathcal{E}^{h,k}]_{C^{1/2}_{[0,1]} L^m} \leq C \left( \|b^k - b\|_{B^{\gamma - 1}_{p}} + (1 + \|b^k\|_{\infty}) n^{-\frac{1}{2} + \varepsilon} + \|b^k\|_{\infty} \|b^k\|_{C^1_{n^{-1+\varepsilon}}} \right).$$
Theoretical Results

Theorem (G.-Haress-Richard)

Assume that \( [u - O]_{C^{3/4}_{[0,1]} L^m, \infty} < \infty \).

(c) **The critical case:** If \( \gamma - 1/p = -1 \) and \( p < +\infty \). If (A2) holds, then there exists two constants \( C_1, C_2 \) that depend on \( m, p, \gamma, \varepsilon, \|b\|_{B^\gamma_p}, \|\psi_0\|_{C_1^{1/2 - \varepsilon}} \) such that for all \( h \in (0,1) \) and \( k \in \mathbb{N} \), the following bound holds:

\[
[\mathcal{E}_{h,k}^{h,k}]_{L^m_{[0,1]} L^m} \leq C_1 \left( \| b - b^k \|_{B^{\gamma-1}_p} (1 + |\log \| b - b^k \|_{B^{\gamma-1}_p}|) + (1 + \| b^k \|_\infty) n^{-\frac{1}{2} + \varepsilon} \right. \\
+ \left. (1 + \| b^k \|_\infty) (1 + \| b^k \|_{C_1}) n^{-1 + \varepsilon} \right)^{C_2}.
\]
Theoretical Results

<table>
<thead>
<tr>
<th>Regularity</th>
<th>$\gamma - \frac{1}{p} = 0$</th>
<th>$\gamma - \frac{1}{p} \in (-1, 0)$</th>
<th>$\gamma - \frac{1}{p} = -1$ and $p &lt; \infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Space Order</strong></td>
<td>$\frac{1}{2} - \varepsilon$</td>
<td>$\frac{1}{2 - 2(\gamma - \frac{1}{p})} - \varepsilon$</td>
<td>$C &gt; 0$</td>
</tr>
<tr>
<td><strong>Time Order</strong></td>
<td>$\frac{1}{4} - \varepsilon$</td>
<td>$\frac{1}{4 - 4(\gamma - \frac{1}{p})} - \varepsilon$</td>
<td>$C/2 &gt; 0$</td>
</tr>
</tbody>
</table>

Table: Rate of convergence of the tamed Euler finite-differences scheme depending on the Besov regularity of the drift.
Numerical simulation - Stochastic heat equation with Dirac drift

Figure: Approximate slope is 0.36
Numerical simulation - regularized Dirac in 0
Numerical simulation - Dirac in 0

Stochastic heat equation with a dirac drift

- t=0.015
- dirac force
Numerical simulation - Dirac in 1

Stochastic heat equation with a dirac drift

- t=0.015
- dirac force
Thanks for your attention!
Regularisation by fractional noise for one-dimensional differential equations with nonnegative distributional drift.

S. Athreya, O. Butkovsky, K. Lê, and L. Mytnik.
Well-posedness of stochastic heat equation with distributional drift and skew stochastic heat equation.

O. Butkovsky, K. Dareiotis, and M. Gerencsér.
Approximation of SDEs: a stochastic sewing approach.

O. Butkovsky, K. Dareiotis, and M. Gerencsér
Optimal rate of convergence for approximations of spdes with non-regular drift.

R. Catellier and M. Gubinelli.
Averaging along irregular curves and regularisation of ODEs.

A. M. Davie.
Uniqueness of solutions of stochastic differential equations.
T. De Angelis, M. Germain, and E. Issoglio.
A numerical scheme for Stochastic Differential Equations with distributional drift.  

F. Delarue and R. Diel.  
Rough paths and 1d SDE with a time dependent distributional drift: application to polymers.  

F. Flandoli, E. Issoglio, and F. Russo.  
Multidimensional stochastic differential equations with distributional drift.  

L. Galeati and M. Gerencsér.  
Solution theory of fractional SDEs in complete subcritical regimes.  

Numerical approximation of fractional SDEs with distributional drift.  

B. Jourdain and S. Menozzi.  
Convergence rate of the Euler-Maruyama scheme applied to diffusion processes with $L^q - L^p$ drift coefficient and additive noise.  
References III

N. V. Krylov and M. Röckner.
Strong solutions of stochastic equations with singular time dependent drift.

K. Lê.
A stochastic sewing lemma and applications.

K. Lê and C. Ling.
Taming singular stochastic differential equations: A numerical method.

J.-F. Le Gall.
One-dimensional stochastic differential equations involving the local times of the unknown process.

T. Nilssen.
Rough linear PDE's with discontinuous coefficients—existence of solutions via regularization by fractional Brownian motion.

D. Nualart and Y. Ouknine.
Regularization of differential equations by fractional noise.