# Explicit Euler scheme: a new way of proof for the existence of solutions for singular parabolic SPDEs

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# SPDE with regular drift

Consider the stochastic reaction-diffusion

$$\partial_t u(t,x) = \frac{1}{2}\partial^2_{xx} u(t,x) + b(u(t,x)) + \xi \tag{$\star$}$$

where $\xi$ is a space-time white noise and $b$ is a (one-sided) Lipschitz function, i.e. polynomial-like function with negative sign of leading coefficient.

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where $\xi$ is a space-time white noise and $b$ is a (one-sided) Lipschitz function, i.e. polynomial-like function with negative sign of leading coefficient.

The existence and uniqueness of solutions are quite well-understood, but these results do not cover the natural cases of discontinuous drifts or reflected diffusions which are expected in physical models.

# SPDE with singular drift

Consider again the stochastic reaction-diffusion

$$\partial_t u(t,x) = \frac{1}{2}\partial^2_{xx} u(t,x) + b(u(t,x)) + \xi \qquad (\star)$$

where $\xi$ is a space-time white noise, but where $b$ is a generalized function in

Besov space $\mathcal{B}^{\gamma}_{p,\infty}(\mathbb{R}^d)$.

In dimension $d = 1$, Athreya, Butkovsky, Lê, Mytnik [ABLM21] proved that there exists a unique strong solution whenever $b$ has some Hölder regularity (precisely Besov regularity $\gamma - 1/p \geq -1$, $\gamma > -1$ and $p \in [1, \infty]$).

# SPDE with singular drift

We can treat the following case

$$\partial_t u(t,x) = \frac{1}{2}\partial^2_{xx} u(t,x) + k\mathbf{1}_{u=0}(u(t,x)) + W_t$$

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So a new way of proof for SPDE with singular drift, reflection, penalization.

Furthermore, without monotonic behavior, or (one-sided) Lipschitz condition.

# Euler scheme

For the SPDE

$$dX_t = AX_t\mathrm{d}t + b(X_t)\mathrm{d}t + \mathrm{d}W_t$$

consider the Euler scheme with time step $h = T/N$

$$X^{n+1} = X^n + A(X^n)h + b(X^n)h + \Delta W^{n+1}$$

then the strong order is given by

$$\mathbb{E}\left[\sup_{0\le n\le N} \|X_{nh} - X^n\|^2\right] \le C_T h.$$

# Euler scheme: rate of convergence

$$\mathbb{E}\left[\sup_{0\leq n\leq N}\|X_{nh}-X^n\|^2\right]\leq C_T h.$$

The usual way of proof is to obtain the bound of moments

$$\mathbb{E}\left[\sup_{0\leq n\leq N}\|X^n\|^p\right]\leq C_T,$$

and it is expected that the order of convergence is given by the time regularity of the noise based on the control of

$$\mathbb{E}\left[\sup_{0\leq t,s\leq T,|t-s|<h}\|W_t-W_s\|^2\right]\leq C_T h \qquad \mathbb{E}\left[\sup_{0\leq t\leq T}\|W_t\|^p\right]\leq C_T,$$

There are many way to obtain the same regularity/moment bounds for the Euler scheme, but it NEEDS an implicit treatment of the differential operator (see also exponential scheme, splitting, etc.). Otherwise it has been shown that the forward/explicit Euler scheme may not be convergent (for instance with superlinear growing of $b$).

## Bounded drift

Consider

$$\begin{aligned} \mathrm{d}X_t &= \mathbb{1}_{\{X_t>0\}}\mathrm{d}t + \mathrm{d}B_t^H, \\ \mathrm{d}X_t &= \mathbb{1}_D^{(d)}(X_t)\mathrm{d}t + \mathrm{d}B_t^H. \end{aligned}$$

Then $b \in \mathcal{C}^{0-}/\mathcal{B}_\infty^0$ leads to $b^n(x) = \sqrt{\frac{n}{2\pi}} \int_0^x e^{-\frac{ny^2}{2}} \mathrm{d}y$ and $n = \lfloor h^{-1} \rfloor$.

## Dirac drift

Consider

$$\mathrm{d}X_t = \delta_0(X_t)\mathrm{d}t + dB_t^H.$$

Then $b \in \mathcal{C}^{-1-}/\mathcal{B}_p^{-d+\frac{d}{p}}$ leads to $b^n(x) = \sqrt{\frac{n}{2\pi}} e^{-\frac{nx^2}{2}}$ and $n = \lfloor h^{-\frac{1}{1+d}} \rfloor$.

Set $h = 2^{-7}10^{-4}$ as reference value.

# Indicator function in dimension 1 - exact algorithm

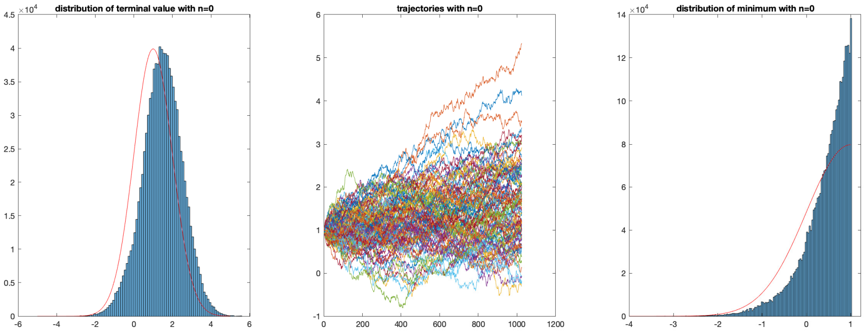



Figure: It works!

# Indicator function in dimension 1 - tamed algorithm

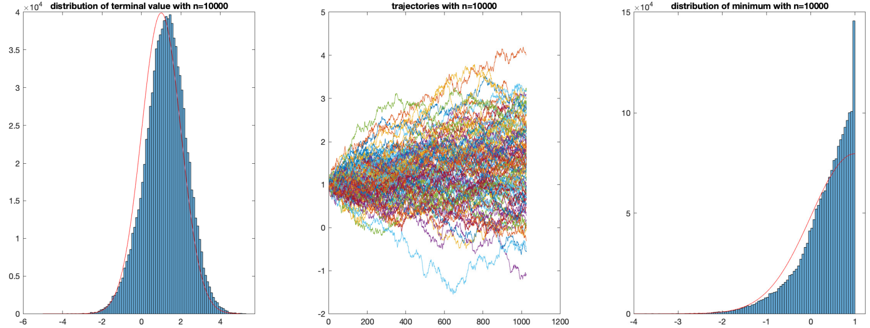



Figure: It works quite well...

# Dirac distribution in dimension 1 - naive algorithm

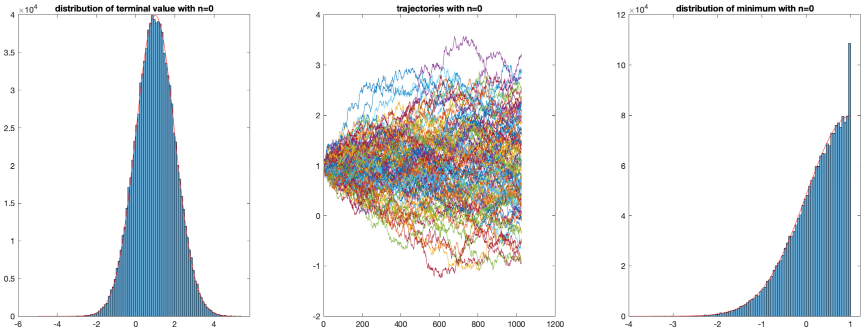
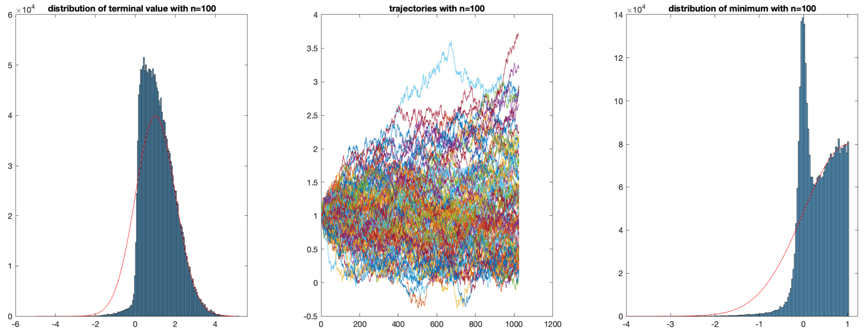



Figure: It does not work!

# Dirac distribution in dimension - tamed algorithm

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# Indicator function in dimension 1

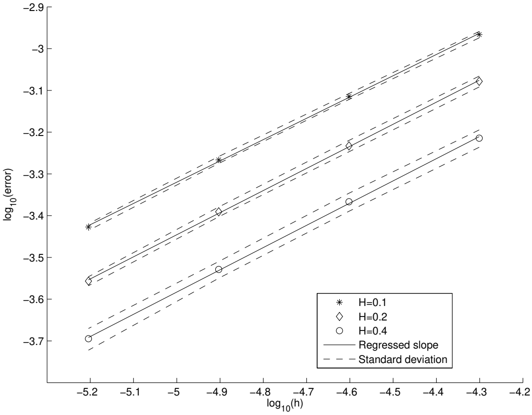


Figure: Approximate slope is 0.5

# Indicator function in dimension 2

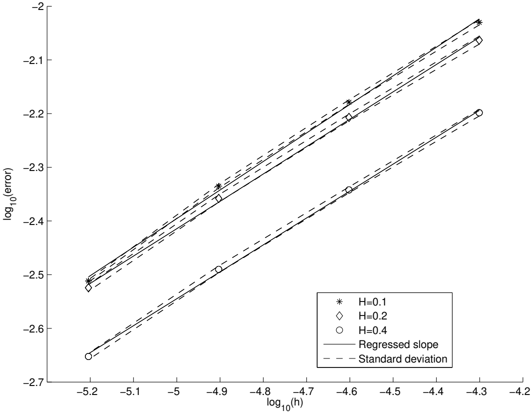


Figure: Approximate slope is 0.5

# Dirac $\delta$ function in dimension 1

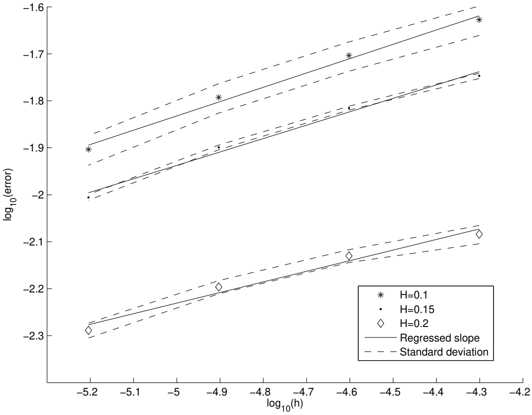


Figure: Approximate slope is 0.25

We study the equation

$$dX_t = b(X_t)dt + dB_t^H,$$

when $b$ is a distribution in some Hölder/Besov space and $B^H$ is a fractional Brownian motion.

We look for solutions of the form

$$X_t = X_0 + K_t + B_t^H,$$

where in case $b$ is regular enough, $K_t = \int_0^t b(X_s)ds$.

**$\mathbb{F}$-fBm**

For a filtration $\mathbb{F}$, we say that $B^H$ is an $\mathbb{F}$-fBm if there exists an $\mathbb{F}$-Brownian motion $W$ s.t. $B_t^H = \int_0^t K_H(t,s)\, dW_s$.

# Weak solutions

$$X_t = X_0 + \int_0^t b(X_s)\,ds + B_t^H, \quad t \in [0, T]. \qquad (*)$$

## Definition

$((X_t)_{t\in[0,T]}, (B_t)_{t\in[0,T]})$ defined on some filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is a weak solution of $(*)$ if

- $B^H$ is an $\mathbb{F}$-fBm;
- $X$ is adapted to $\mathbb{F}$;
- $\exists (K_t)_{t\in[0,T]}$ such that, a.s.,

$$X_t = X_0 + K_t + B_t^H, \quad \forall t \in [0, T];$$

- $\forall (b^k)_{k\in\mathbb{N}}$ smooth bounded functions converging to $b$ in $\mathcal{C}^\alpha / \mathcal{B}_p^{\gamma-}$,

$$\sup_{t\in[0,T]} \left| \int_0^t b^k(X_r)dr - K_t \right| \xrightarrow[k\to+\infty]{\mathbb{P}} 0.$$

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- $\mathbb{1}_{\mathbb{R}_+} \in \mathcal{B}^{0}_{\infty,\infty}(\mathbb{R}^1)$.
- For $\gamma - d/p < 0$, the space $\mathcal{B}^{\gamma}_{p,\infty}(\mathbb{R}^d)$ contains genuine distributions;
- $\delta_0 \in \mathcal{B}^{0}_{1,\infty}(\mathbb{R})$ (or $\mathcal{B}^{-d+\frac{d}{p}}_{p,\infty}(\mathbb{R}^d)$ in dimension $d$).

# Weak and strong existence

## Theorem ([Anzeletti-Richard-Tanré '21], [G.-Haress-Richard.'22])

*Let* $\gamma \in \mathbb{R}$, $p \in [1, \infty]$, $b \in \mathcal{C}^\alpha / \mathcal{B}_p^\gamma$.
*[weak] Assume that*

$$\alpha = \gamma - \frac{d}{p} > \frac{1}{2} - \frac{1}{2H}.$$

*Then there exists a weak solution* $X$ *s.t.* $X - B^H \in \mathcal{C}^\kappa_{[0,T]}(L^m)$, *for all* $m \geq 2$,

$$\forall \kappa \in (0, 1 + H\alpha] \setminus \{1\}, \qquad \forall \kappa \in \left(0, 1 + H\left(\gamma - \frac{d}{p}\right)\right] \setminus \{1\}.$$

*[strong] Assume that*

$$H < \frac{1}{2}, \ \gamma - \frac{d}{p} < 0 \text{ and } \gamma - \frac{d}{p} > 1 - \frac{1}{2H}.$$

*Then there exists a strong solution* $X$ *to* $(*)$ *such that* $X - B \in \mathcal{C}^{\frac{1}{2}+H}_{[0,T]}(L^m)$ *for any* $m \geq 2$. *Besides, pathwise uniqueness holds in the class of all solutions* $X$ *such that* $X - B \in \mathcal{C}^{\frac{1}{2}+H}_{[0,T]}(L^2)$.

# Reflection?

## Example

If $b = \delta_0$ ($b \in \mathcal{C}^{-1}/\mathcal{B}^0_1$) and $d = 1$, one must choose $H < \frac{1}{3}$.
Then $X - B^H$ has Hölder regularity $1 - H$ ($> H$), hence $X$ is not reflected.

# SPDE with singular drift

We conclude the introduction by commenting on the needs to treat SDE first.

It is known that for each fixed space point, the free stochastic heat equation (that is, equation $(\star)$ with $b = 0$) behaves "qualitatively" like a fractional Brownian motion (fBM) with the Hurst parameter $H = 1/4$.

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Indeed, the solution of

$$\partial_t u(t,x) = \frac{1}{2}\partial^2_{xx} u(t,x) + \dot{W}(t,x)$$

is given by the well-known stochastic convolution

$$Z(t,x) = e^{\frac{t}{2}\Delta} u_0(x) + \int_0^t e^{\frac{t-s}{2}\Delta} \mathrm{d}W(s,x)$$

which has regularity $\mathcal{C}^{\frac{1}{4}-}$ in time and $\mathcal{C}^{\frac{1}{2}-}$ in space.

# SPDE with singular drift towards SDE with fractional noise

To solve

$$\partial_t u(t,x) = \frac{1}{2}\partial^2_{xx} u(t,x) + b(u(t,x)) + \dot{W}(t,x). \qquad (\star)$$

Denote $Y(t,x) = u(t,x) - Z(t,x)$, then the previous remark leads to solve

$$\partial_t (u(t,x) - Z(t,x)) = \frac{1}{2}\partial^2_{xx}(u(t,x) - Z(t,x)) + b(u(t,x) - Z(t,x) + Z(t,x))$$

a random PDE with $\mathcal{C}^{\frac{1}{4}-}$ regularity in translation.

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$$\partial_t u(t,x) = \frac{1}{2}\partial_{xx}^2 u(t,x) + b(u(t,x)) + \dot{W}(t,x). \qquad (\star)$$

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That is $b \in \mathcal{C}^\alpha/\mathcal{B}_p^\gamma$, where $\alpha = \gamma - 1/p > -1$.
Note that the Dirac delta function lies in $\mathcal{B}_p^{-1+1/p}$ which is the critical case.

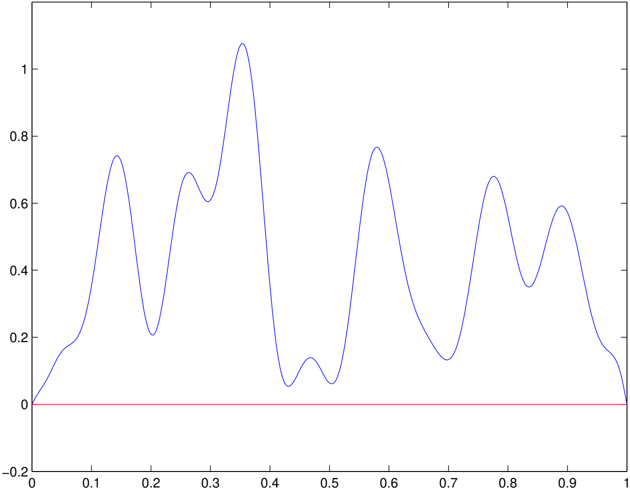


Figure: Before touching

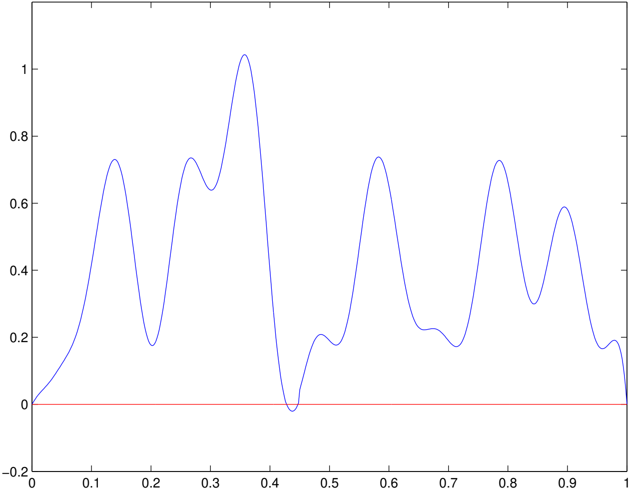


Figure: Breaks physical limit

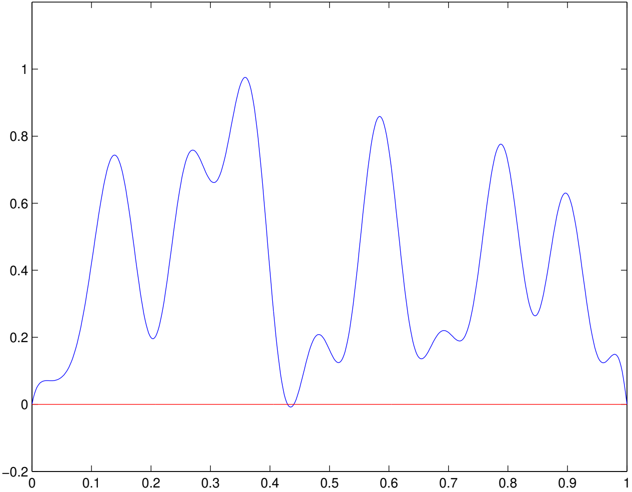


Figure: Penalization steps

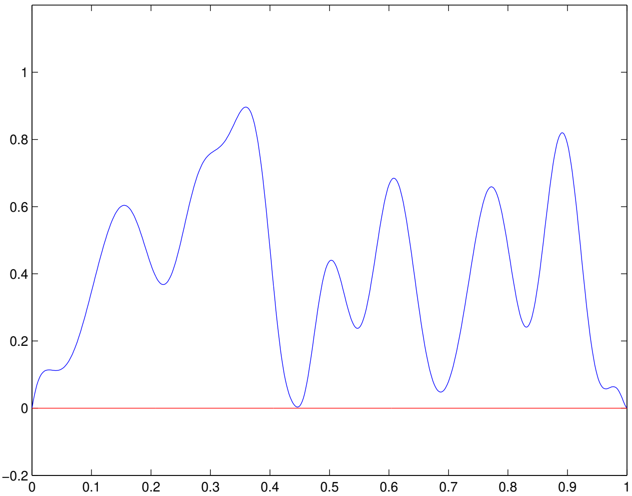


Figure: Reflection

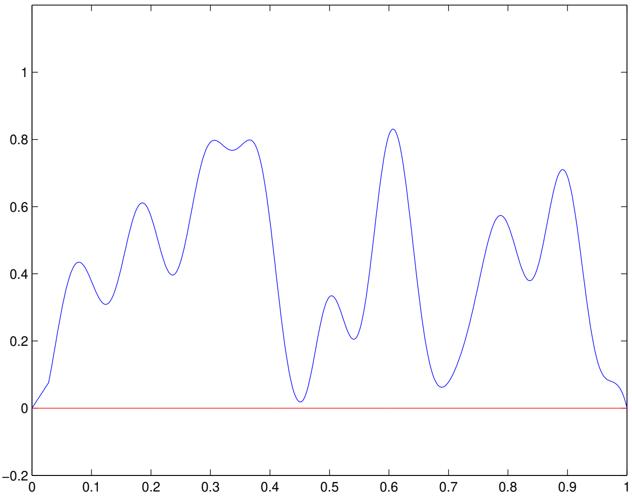


Figure: Leaves physical limit

# Elements of proof: The basic ingredient is the *stochastic* sewing lemma.

**Lemma ([Lê'20])**

*Let* $m \in [2, \infty)$. *Let* $A : \Delta_{0,1} \to L^m(\Omega)$ *s.t.* $A_{s,t}$ *is* $\mathcal{F}_t$*-measurable. Assume* $\exists \Gamma_1, \Gamma_2 \geq 0$, *and* $\varepsilon_1, \varepsilon_2 > 0$ *s.t.* $\forall (s,t) \in \Delta_{0,1}$ *and* $u := \frac{s+t}{2}$,

$$\|\mathbb{E}^s[\delta A_{s,u,t}]\|_{L^m} \leq \Gamma_1 (t-s)^{1+\varepsilon_1},$$
$$\|\delta A_{s,u,t}\|_{L^m} \leq \Gamma_2 (t-s)^{\frac{1}{2}+\varepsilon_2}.$$

*Then* $\exists (\mathcal{A}_t)_{t\in[0,1]}$ *s.t.* $\forall t \in [0,1]$ *and any sequence of partitions* $\Pi_k = \{t_i^k\}_{i=0}^{N_k}$ *of* $[0,t]$ *with mesh size going to zero,*

$$\mathcal{A}_t = \lim_{k\to\infty} \sum_{i=0}^{N_k} A_{t_i^k, t_{i+1}^k} \text{ in proba.}$$

*Moreover,* $\exists C$ *s.t.* $\forall (s,t) \in \Delta_{0,1}$,

$$\|\mathcal{A}_t - \mathcal{A}_s - A_{s,t}\|_{L^m} \leq C\,\Gamma_1 (t-s)^{1+\varepsilon_1} + C\,\Gamma_2 (t-s)^{\frac{1}{2}+\varepsilon_2},$$
$$\|\mathbb{E}^s[\mathcal{A}_t - \mathcal{A}_s - A_{s,t}]\|_{L^m} \leq C\,\Gamma_1 (t-s)^{1+\varepsilon_1}.$$

# Elements of proof

It leads to key estimates for "smooth" $f$:

- $(\psi_t)$ is $\mathbb{F}$-adapted, $m \in [2, \infty)$, $p \in [m, \infty]$ and $\gamma < 0$ s.t. $(\gamma - d/p) > -1/(2H)$. Let $\alpha \in (0,1)$ s.t. $H(\gamma - d/p - 1) + \alpha > 0$,

$$\Big\| \int_s^t f(B_r + \psi_r)\, dr \Big\|_{L^m(\Omega)} \leq C\, \|f\|_{\mathcal{B}_p^\gamma} (t-s)^{1+H(\gamma - d/p)} + C\, \|f\|_{\mathcal{B}_p^\gamma} [\psi]_{\mathcal{C}^\alpha_{[s,t]} L^m} (t-s)^{1+H(\gamma-d/p-1)+\alpha}. \tag{1}$$

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- $m \in [2,\infty)$ and $p \in [m,\infty]$ and $0 > \gamma > 1 - 1/(2H)$. For any $\mathcal{F}_s$-measurable $\kappa_1, \kappa_2 \in L^m(\Omega)$,

$$\Big\| \int_s^t f(B_r + \kappa_1) - f(B_r + \kappa_2) dr \Big\|_{L^m} \leq C \|f\|_{\mathcal{B}_p^\gamma} \|\kappa_1 - \kappa_2\|_{L^m} (t-s)^{1+H(\gamma-d/p-1)}, \quad (2)$$

  (for uniqueness).

# Elements of proof II

*Why the condition* $\gamma - \frac{d}{p} > \frac{1}{2} - \frac{1}{2H}$ *in the theorem?*

Consider $(b^n)$ in $\mathcal{C}^\infty \cap \mathcal{B}_p^\gamma$ that approximates $b$. Let $X^n$ be the solution with drift $b^n$.

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2. In (1), replace $f$ by $b^n$, $X$ by $X^n$ and $K_t^n := \int_0^t b^n(X_s)ds \equiv \psi_t$. Then for any $\alpha$ such that $H(\gamma - d/p - 1) + \alpha > 0$,

$$\begin{aligned}\|K_t^n - K_s^n\|_{L^m(\Omega)} \leq & C\, \|b^n\|_{\mathcal{B}_p^\gamma} (t-s)^{1+H(\gamma-d/p)} \\ & + C\, \|b^n\|_{\mathcal{B}_p^\gamma} [K^n]_{\mathcal{C}^\alpha_{[s,t]} L^m} (t-s)^{1+H(\gamma-d/p-1)+\alpha}.\end{aligned}$$

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$$\|K_t^n - K_s^n\|_{L^m(\Omega)} \leq C \|b^n\|_{\mathcal{B}_p^\gamma} (t-s)^{1+H(\gamma - d/p)} + C \|b^n\|_{\mathcal{B}_p^\gamma} [K^n]_{\mathcal{C}^\alpha_{[s,t]} L^m} (t-s)^{1+H(\gamma-d/p-1)+\alpha}.$$
3. Choosing $\alpha = 1 + H(\gamma - d/p)$ above with $H(\gamma - d/p - 1) + \alpha > 0$ requires $\gamma - \frac{d}{p} > \frac{1}{2} - \frac{1}{2H}$. Then for $t-s$ small enough,
$$[K^n]_{\mathcal{C}^\alpha_{[s,t]} L^m} \leq C \|b^n\|_{\mathcal{B}_p^\gamma} + \frac{1}{2} [K^n]_{\mathcal{C}^\alpha_{[s,t]} L^m}.$$

# Elements of proof II

*Why the condition* $\gamma - \frac{d}{p} > \frac{1}{2} - \frac{1}{2H}$ *in the theorem?*

Consider $(b^n)$ in $\mathcal{C}^\infty \cap \mathcal{B}_p^\gamma$ that approximates $b$. Let $X^n$ be the solution with drift $b^n$.

1. We look for *a priori* estimate on the Hölder regularity of $X^n - B$. If uniform in $n$, we get tightness.
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$$\begin{aligned}\|K_t^n - K_s^n\|_{L^m(\Omega)} \leq & C\,\|b^n\|_{\mathcal{B}_p^\gamma}(t-s)^{1+H(\gamma-d/p)} \\ & + C\,\|b^n\|_{\mathcal{B}_p^\gamma}[K^n]_{\mathcal{C}^\alpha_{[s,t]}L^m}(t-s)^{1+H(\gamma-d/p-1)+\alpha}.\end{aligned}$$
3. Choosing $\alpha = 1 + H(\gamma - d/p)$ above with $H(\gamma - d/p - 1) + \alpha > 0$ requires $\gamma - \frac{d}{p} > \frac{1}{2} - \frac{1}{2H}$. Then for $t - s$ small enough,
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4. Extend to any $s \leq t$ and get tightness.

Let $h > 0$ and $(b^n)$ that approximates $b$ in $\mathcal{B}_p^{\gamma-}(\mathbb{R}^d)$. Consider the following tamed Euler scheme:

$$X_t^{h,n} = X_0 + \int_0^t b^n(X_{r_h}^{h,n})dr + B_t,$$

where $r_h = h\lfloor \frac{r}{h} \rfloor$.

# "Subcritical" case

**Theorem ([G.-Haress-Richard.'22])**

*Let* $H < \frac{1}{2}$, $m \geq 2$ *and* $p \in [m, \infty]$. *Let* $b \in \mathcal{B}_p^\gamma$ *and assume*

$$0 > \gamma - \frac{d}{p} > 1 - \frac{1}{2H}.$$

*Let* $X$ *denote a weak solution such that* $X - B \in \mathcal{C}_{[0,T]}^{\frac{1}{2}+H}(L^m)$. *Let* $\varepsilon \in (0, \frac{1}{2})$. *Then* $\forall h \in (0,1)$ *and* $\forall n \in \mathbb{N}$,

$$[X_t - X_t^{h,n}]_{\mathcal{C}^{\frac{1}{2}} L^m} \leq C\Big(\|b^n - b\|_{\mathcal{B}_p^{\gamma-1}} + \|b^n\|_\infty h^{\frac{1}{2}-\varepsilon} + \|b^n\|_\infty \|b^n\|_{\mathcal{C}^1} h^{1-\varepsilon}\Big).$$

Denote by $G_t$ the Gaussian semigroup with variance $t$. Then

$$\|G_{\frac{1}{n}} b - b\|_{\mathcal{B}_p^{\gamma-1}} \lesssim n^{-\frac{1}{2}} \|b\|_{\mathcal{B}_p^{\gamma}},$$

$$\|G_{\frac{1}{n}} b\|_{\infty} \lesssim n^{\frac{1}{2}(-\gamma+\frac{d}{p})} \|b\|_{\mathcal{B}_p^{\gamma}},$$

$$\|G_{\frac{1}{n}} b\|_{\mathcal{C}^1} \lesssim n^{\frac{1}{2}(1-\gamma+\frac{d}{p})} \|b\|_{\mathcal{B}_p^{\gamma}}.$$

**Corollary**

*Let $h \in (0, \frac{1}{2})$, $n_h = \lfloor h^{-\frac{1}{1-\gamma+\frac{d}{p}}} \rfloor$ and $b^{n_h} = G_{\frac{1}{n_h}} b$. Then*

$$[X_t - X_t^{h,n_h}]_{\mathcal{C}^{\frac{1}{2}} L^m} \leq C\, h^{\frac{1}{2(1-\gamma+\frac{d}{p})} - \varepsilon}.$$

# "Critical" case

**Theorem ([G.-Haress-Richard.'22])**

*Let* $H < \frac{1}{2}$, $m \geq 2$ *and* $p \in [m, \infty]$. *Let* $b \in \mathcal{B}_p^\gamma$ *and assume*

$$\gamma - \frac{d}{p} = 1 - \frac{1}{2H} \quad \text{and} \quad p < +\infty.$$

*Let*

$$\epsilon(h,n) = \|b - b^n\|_{\mathcal{B}_p^{\gamma-1}}(1 + |\log(\|b - b^n\|_{\mathcal{B}_p^{\gamma-1}})|) + \|b^n\|_\infty h^{\frac{1}{2}-\varepsilon} + \|b^n\|_{\mathcal{C}^1}\|b^n\|_\infty h^{1-\varepsilon}.$$

*Then* $\exists C, \rho > 0$ *such that for all* $h \in (0,1)$ *and* $n \in \mathbb{N}$,

$$[X_t - X_t^{h,n}]_{\mathcal{C}^{\frac{1}{2}}-L^m} \leq C\epsilon(h,n)^\rho.$$

# Rates

The orders of convergence obtained here compared to regular $b$ from Butkovsky et al. (2021a); Dareiotis at al. (2021); De Angelis et Al. (2019) are:

| *Drift* | *Rate* |
|---|---|
| $\alpha > 0$ | $\left(\frac{1}{2} + H\alpha\right) \wedge 1$ |
| $\gamma - \frac{d}{p} > 0$ | $\left(\frac{1}{2} + H\left(\gamma - \frac{d}{p}\right)\right) \wedge 1 - \varepsilon$ |
| $\alpha = 0 \sim$ Bounded | $\frac{1}{2}$ |
| $\gamma - \frac{d}{p} = 0$ | $\frac{1}{2} - \varepsilon$ |
| $\gamma - \frac{d}{p} \in \left(1 - \frac{1}{2H}, 0\right)$ | $\frac{1}{2(1 - \gamma + \frac{d}{p})} - \varepsilon$ |
| $\gamma - \frac{d}{p} = 1 - \frac{1}{2H}$ and $p < \infty$ | $\rho = He^{-\mathrm{M}} - \varepsilon$ |

# Elements of proof I

Define

$$K_t^n := \int_0^t b^n(X_r)dr \text{ and } K_t^{h,n} := \int_0^t b^n(X_{r_h}^{h,n})dr.$$

Decompose the error as follows:

$$X_t - X_t^{h,n} - (X_s - X_s^{h,n}) = K_t - K_s - (K_t^n - K_s^n) \tag{3}$$

$$+ \int_s^t (b^n(K_r + B_r) - b^n(K_r^{h,n} + B_r))dr \tag{4}$$

$$+ \int_s^t (b^n(K_r^{h,n} + B_r) - b^n(K_{r_h}^{h,n} + B_{r_h}))dr. \tag{5}$$

Denote by $E_{s,t}^{1,h,n}$ the term in (4), $E_{s,t}^{2,h,n}$ the term in (5), and $E_{s,t}^{h,n} = E_{s,t}^{1,h,n} + E_{s,t}^{2,h,n}$.

# Elements of proof II

The terms (3) and (4) are controlled "classically" by stochastic sewing:

$$[K - K^n]_{\mathcal{C}^{\frac{1}{2}}_{[s,t]} L^m} \leq C \|b - b^n\|_{\mathcal{B}^{\gamma-1}_p}$$

and

$$\|E^{1,h,n}_{s,t}\|_{L^m} \leq C([E^{h,n}]_{\mathcal{C}^{\frac{1}{2}}_{[s,t]} L^m} + \|b - b^n\|_{\mathcal{B}^{\gamma-1}_p})(t-s)^{1+H(\gamma - \frac{d}{p} - 1)}.$$

# Elements of proof II

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and

$$\|E^{1,h,n}_{s,t}\|_{L^m} \leq C([E^{h,n}]_{\mathcal{C}^{\frac{1}{2}}_{[s,t]} L^m} + \|b - b^n\|_{\mathcal{B}_p^{\gamma-1}})(t-s)^{1+H(\gamma - \frac{d}{p} - 1)}.$$

The last term $E^{2,h,n}_{s,t}$ can be controlled using Girsanov, but this leads to an exponential dependence in $\|b^n\|_\infty$. Instead with use again the SSL to get

$$\begin{aligned}\|E^{2,h,n}_{s,t}\|_{L^m} \leq C\Big( & (\|b^n\|_\infty + 1)\, h^{\frac{1}{2}}\, (t-s)^{1+H(\gamma - \frac{d}{p} - 1)} \\ & + \|b^n\|_\infty\, h^{\frac{1}{2} - \varepsilon}\, |t-s|^{\frac{1}{2} + \varepsilon} \\ & + \|b^n\|_{\mathcal{C}^1}\, \|b^n\|_\infty\, h^{1-\varepsilon}\, (t-s)^{1+\varepsilon}\Big).\end{aligned}$$

# Consequences

## Definition

- *Pathwise uniqueness* holds if for any solutions $(X, B)$ and $(Y, B)$ defined on the same filtered probability space with the same $B$ and same initial condition $X_0$, $X$ and $Y$ are indistinguishable.
- A weak solution $(X, B)$ such that $X$ is $\mathbb{F}^B$-adapted is called a *strong solution*.

# Strong existence and uniqueness

As a consequence of the convergence of the Euler scheme,

**Theorem ([G.-Haress-Richard.'22])**

*Let $H < \frac{1}{2}$, $\gamma \in \mathbb{R}$, $p \in [1, \infty]$ and $b \in \mathcal{B}_p^\gamma$. Assume that*

$$0 > \gamma - \frac{d}{p} \geq 1 - \frac{1}{2H} \quad \text{and} \quad \gamma > 1 - \frac{1}{2H}.$$

- *There exists a strong solution $X$ to $(*)$ such that $[X - B]_{\mathcal{C}^{1/2+H}_{[0,1]} L^{m,\infty}} < \infty$ for any $m \geq 2$.*
- *Pathwise uniqueness holds in the class of solutions such that $[X - B]_{\mathcal{C}^{1/2+H}_{[0,1]} L^{2,\infty}} < \infty$.*

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- *There exists a strong solution $X$ to $(*)$ such that $[X - B]_{\mathcal{C}^{1/2+H}_{[0,1]} L^{m,\infty}} < \infty$ for any $m \geq 2$.*
- *Pathwise uniqueness holds in the class of solutions such that $[X - B]_{\mathcal{C}^{1/2+H}_{[0,1]} L^{2,\infty}} < \infty$.*
- *If $\gamma - \frac{d}{p} > 1 - \frac{1}{2H}$, for all $\eta \in (0,1)$, pathwise uniqueness holds in the class of solutions such that $[X - B]_{\mathcal{C}^{H(1-\gamma+d/p)+\eta}_{[0,1]} L^{2,\infty}} < \infty$.*

1. Singular model of equations
   - Numerical simulations
   - SDE Weak solutions
2. Sewing approach and Tamed Euler scheme
   - Sewing approach
   - Tamed Euler scheme
   - Strong existence and uniqueness of the SDE
3. Tamed Euler scheme for SPDE
   - Finite differences
   - Controlling the error

## Stochastic PDEs

Consider

$$\partial_t u(t,x) = \frac{1}{2}\partial^2_{xx} u(t,x) + b(u(t,x)) + \xi \qquad (\star)$$

where $\xi$ is a space-time white noise and $b$ is a distribution in some Besov space, with bounded mesurable initial data $\psi_0$.

In [Athreya et al.'20], it is proven that there exists a weak solution with certain regularity. The goal is to approximate it.

# Stochastic PDEs

Consider

$$\partial_t u(t,x) = \frac{1}{2}\partial_{xx}^2 u(t,x) + b(u(t,x)) + \xi \qquad (\star)$$

where $\xi$ is a space-time white noise and $b$ is a distribution in some Besov space, with bounded mesurable initial data $\psi_0$.

In [Athreya et al.'20], it is proven that there exists a weak solution with certain regularity. The goal is to approximate it.

Consider a sequence $b^k$ of smooth function which approximates the distribution $b$ in some sense.
We say that a sequence of smooth bounded functions $(b^k)$ converges to $b$ in $\mathcal{B}_p^{\gamma-}$ as $k$ goes to infinity if

$$\begin{cases} \sup_{k\in\mathbb{N}} \|b^k\|_{\mathcal{B}_p^\gamma} < \|b\|_{\mathcal{B}_p^\gamma} < \infty \\ \lim_{k\to\infty} \|b^k - b\|_{\mathcal{B}_p^{\gamma'}} = 0 \qquad \forall \gamma' < \gamma. \end{cases}$$

# Definition of solution

## Definition

A couple $\left((u_t(x))_{\substack{t\in[0,1]\\x\in[0,1]}}, \xi\right)$ is a weak solution on some filtered space $\left(\Omega, \mathbb{P}, \mathbb{F}\right)$ if there exists a process $K : [0,1] \times [0,1] \times \Omega \to \mathbb{R}$ such that

(1) $\xi$ is an $\mathbb{F}$-space time white noise.

(2) $u$ is adapted to $\mathbb{F}$.

(3) $u_t(x) = P_t\psi_0(x) + K_t(x) + O_t(x)$ a.s where $x \in [0,1]$, $t \in [0,1]$.

(4) For any sequence $(b^k)_{k\in\mathbb{N}}$ in $\mathcal{C}^b_\infty$ converging to $b$ in $B^{\gamma-}_p$, we have

$$\sup_{\substack{t\in[0,1]\\x\in[0,1]}} \left| \int_0^t \int_0^1 p_{t-r}(x,y) b^k(u_r(y)) \mathrm{d}y\mathrm{d}r - K_t(x) \right| \xrightarrow[k\to\infty]{\mathbb{P}} 0.$$

(5) Almost surely, the function $u$ is continuous on $[0,1] \times [0,1]$.

If the couple is clear from the context, we simply say that $u$ is a weak solution.

# Mild form and Gaussian operators

The *mild* form associated to the SPDE is

$$u_t(x) = P_t\psi_0(x) + \int_0^t \int_0^1 p_{t-r}(x,y)b(u_r(y))\mathrm{d}y\mathrm{d}r + O_t(x) \ .$$

Notice that the *mild* form is also not well-posed when b is a genuine distribution. To define a solution , we first specify in which spaces we take $b$, and then define how we approximate $b$ via a smooth sequence.

# Mild form and Gaussian operators

We encounter different heat kernels: the continuum Gaussian on $\mathbb{R}$, $g_t(x) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right)$, the continuum heat kernel on $[0,1]$ associated with the boundary conditions

$$p_t(x,y) = p_t(x-y) = \sum_{k\in\mathbb{Z}} \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{(x-y+k)^2}{4t}\right) = \sum_{k\in\mathbb{Z}} e^{-4\pi^2 k^2 t} e^{i2\pi k(x-y)}.$$

We denote by $G_\cdot$ and $P_\cdot$ the respective convolutions with the $g_\cdot$ and $p_\cdot$.

That is, for any bounded measurable function $f$, we write

$$G_t f(x) = \int_0^1 g_t(x-y) f(y) dy, \quad \text{and} \quad P_t f(x) = \int_0^1 p_t(x-y) dy.$$

We also denote the convolution operator by $(f \star g)(x) = \int f(x-y) g(y) \mathrm{d}y$.

# Mild form and Gaussian operators

We denote by $Q(t)$ the variance of the Ornstein-Uhlenbeck process $O_t(x) = \int_0^t \int_0^1 p_{t-r}(x,y)\xi(\mathrm{d}y, \mathrm{d}r)$. In fact, for all $x \in [0,1]$

$$\mathbb{E}\Big(O_t(x)^2\Big) = \int_0^t \int_0^1 p_t(x-y)^2 \mathrm{d}y\mathrm{d}r = \int_0^t \int_0^1 p_{2t}(y)\mathrm{d}y\mathrm{d}r =: Q(t).$$

Moreover, writing $Z_{s,t}(x) = O_t(x) - P_{t-s}O_s(x)$ for $t \geq s$, we have that the random variable $Z_{s,t}(x)$ is independent of $\mathcal{F}_s$ and

$$\mathbb{E}\Big(Z_{s,t}(x)\Big)^2 = Q(t-s).$$

The following equality will be used a lot. For any continuous function $h$ and $\mathcal{F}_s$-measurable random variable $Y$ one has the almost sure equality

$$\mathbb{E}^s h\left(O_t(x) + Y\right) = G_{Q(t-s)} h\left(P_{t-s}O_s(x) + Y\right).$$

# Finite differences

We study a tamed Euler finite-difference.

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Let $h \in (0,1)$ be a time step of the form $h = c(2n)^{-2}$ for some $n \in \mathbb{N}$, where $c$ is a constant satisfying the CFL condition $c > \frac{1}{2}$.

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We introduce the following space and time grids.

$$\Pi_n = \left\{0, (2n)^{-1}, \dots, (2n-1)(2n)^{-1}\right\}, \quad \Lambda_h = \{0, h, 2h, \dots\}.$$

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$$\Pi_n = \left\{0, (2n)^{-1}, \ldots, (2n-1)(2n)^{-1}\right\}, \quad \Lambda_h = \{0, h, 2h, \ldots\}.$$

We can now define the numerical scheme for $x \in \Pi_n$ and $t \in \Lambda_h$ as

$$\begin{cases} u_{t+h}^{h,k}(x) = u_t^{h,k}(x) + h\Delta_n u_t^{h,k}(x) + hb^k\left(u_t^{h,k}(x)\right) + h\eta_{h,n}(t,x) \\ u_0^{h,k}(x) = \psi_0(x), \end{cases}$$

# Finite differences

$\Delta_n$ is the discrete Laplacian

$$\Delta_n f(x) = (2n)^2 \left(f\left(x + (2n)^{-1}\right) - 2f(x) + f\left(x - (2n)^{-1}\right)\right),$$

and discrete noise $\eta_{h,n}$ is given by

$$\eta_h(t,x) = (2n)h^{-1}\xi\left([t, t+h] \times \left[x, x + (2n)^{-1}\right]\right).$$

# The goal is the rate of convergence

## Rate of convergence

For the right choice of $b^k$ and $k$, we prove the following rate of convergence for the approximation $u^{h,k}$

$$\sup_{\substack{t\in\Lambda_h \\ x\in\Pi_n}} ||u_t - u_t^{h,k}||_{L^m(\Omega)} \leq C\left(n^{-\frac{1}{2(1-\gamma+\frac{1}{p})}}\right).$$

# Approximated operator

Consider the functions $e_j(x) = e^{i2\pi jx}$ for $j \in \mathbb{Z}$. They are eigenfunctions of $\Delta$ with eigenvalues $\lambda_j = -4\pi^2 j^2$. It is well known that $(e_j)_{j\in\mathbb{Z}}$ forms an orthonormal basis of $L^2([0,1],\mathbb{C})$.

The eigenvalues of the discrete Laplacian $\Delta_h$ are

$$\lambda_j^n = -16n^2 \sin^2\left(\frac{j\pi}{2n}\right) \text{ for j } \in \mathbb{Z} \, .$$

# Approximated operator

We know that for $-n \leq j, l \leq n-1$,

$$\Delta_h e_j(x) = \lambda_j^h e_j(x),$$

and

$$\frac{1}{2n} \sum_{x \in \Pi_n} e_j(x) \overline{e_\ell(x)} = 1_{j=\ell}.$$

As a consequence, $e_i$ for $i \in [-n, n-1]$, as functions on $\Pi_n$ form a basis of $L^2(\Pi_n; \mathbb{C})$.

It will be convenient to use the piecewise linear extension of the restriction of $e_j$ to $\Pi_n$ : for $-n \leq j \leq n-1$, for $x \in \Pi_n$, and $x' \in \left[x, x + (2n)^{-1}\right]$, set

$$e_j\left(x'\right) = e_j(x) + (2n)\left(x' - x\right)\left(e_j\left(x + (2n)^{-1}\right) - e_j(x)\right).$$

# Approximated operator

It remains to encode the temporal discretisation.

Naturally, on the temporal gridpoints $t = kh$ a factor $\left(1 + h\lambda_j^n\right)^k$ appears.

Between the gridpoints, we again interpolate linearly. More precisely, for $j = -n, \ldots, n-1$, for $t \in \Lambda_h$, and $t' \in [t, t+h]$, set

$$\mu_j^h\left(t'\right) = \left(1 + h\lambda_j^n\right)^{th^{-1}} + h^{-1}\left(t' - t\right)\left(\left(1 + h\lambda_j^n\right)^{(t+h)h^{-1}} - \left(1 + h\lambda_j^n\right)^{th^{-1}}\right).$$

# Discrete mild form

We can now define the discrete heat kernel and discrete convolution.
Our goal is to write the numerical scheme in a *mild* form similar to classical mild form for SPDEs.

For $t \in [0,1]$ and $x \in [0,1]$, denote by $t_h$ and $x_n$ the leftmost gridpoint from $t$ in $\Lambda_h$ and from $x$ in $\Pi_n$ respectively.

The *mild* form associated to the scheme is

$$u_t^{h,k}(x) = P_t^n \psi_0(x) + \int_0^t P_{(t-s)_h}^n b^k\left(u_{s_h}^{h,k}\right)(x)\mathrm{d}s + \int_0^t \int_0^1 p_{(t-s)_h}^n(x,y)\xi(\mathrm{d}y,\mathrm{d}s)$$

where $P^n$ and $p^n$ are respectively the discrete analogues of $P$ and $p$.

# Discrete mild form

They are defined for all $t \in [0,1]$ and $x \in [0,1]$

$$P_t^n f(x) := p_t^n \star_n f \text{ and } p_t^n(x,y) = \sum_{j=-n}^{n-1} \mu_j^n(t) e_j^n(x) \overline{e_j^n(y_n)}.$$

Moreover, $\star_n$ denotes the discrete convolution

$$g \star_n f(x) := \int_0^1 g(x-y) f(y_n) \mathrm{d}y.$$

# Ornstein-Uhlenbeck process

## Definition

We define the discrete Ornstein-Uhlenbeck process as

$$O_t^n(x) := \int_0^t \int_0^1 p_{(t-r)_h}^h(x,y)\xi(\mathrm{d}y, \mathrm{d}r) \ ,$$

Analogously to $O_t$, for $s \leq t$, we define $\widehat{O}_{s,t}^h$ and $Z_{s,t}^h$ by

$$\begin{aligned} O_t^n(x) &= \int_0^s \int_0^1 p_{(t-r)_h}^n(x,y)\xi(dy, dr) + \int_s^t \int_0^1 p_{(t-r)_n}^n(x,y)\xi(\mathrm{d}y, \mathrm{d}r) \\ &=: \widehat{O}_{s,t}^n + Z_{s,t}^n. \end{aligned}$$

# Ornstein-Uhlenbeck process

Notice that $Z^n_{s,t}(x)$ is a random variable independent of $\mathcal{F}_s$ and has variance

$$\mathbb{E}\left(Z^n_{s,t}(x)^2\right) = \int_0^{t-s}\int_0^1 |p^n_{r_n}(x,y)|^2\,\mathrm{d}y\mathrm{d}r$$
$$= \int_0^{t-s}\sum_{j=-n}^{n-1}\left|1+h\lambda^n_j\right|^{2r_h h^{-1}}\mathrm{d}r =: Q^n(t-s)\;.$$

Also similarly to $O_t$, we have that for any continuous function $h$ and $\mathcal{F}_s$-measurable random variable $Y$

$$\mathbb{E}^s h\left(O^n_t(x)+Y\right) = G_{Q^n(t-s)}h\left(\widehat{O}^n_{s,t}(x)+Y\right)\;.$$

# Controlling the error

We are interested in controlling the error $\mathcal{E}^{h,k}$ between the solution $u$ and the numerical scheme $u^{h,k}$, defined for $(s,t) \in \Delta_{0,1}$ and $x \in [0,1]$ by

$$\mathcal{E}^{h,k}_{s,t}(x) = u_t(x) - P_{t-s}u_s(x) - \int_s^t P^n_{(t-s)_h} b^k(u^{h,k}_{r_h})(x)dr.$$

# Elements of Proof I

For all $(s,t) \in \Delta_{0,1}$ and $x \in [0,1]$, we write

$$\begin{aligned}
\mathcal{E}^{h,k}_{s,t}(x) &= v_t(x) - P_{t-s}v_s - \int_s^t P^n_{(t-r)_h} b^k(v^{h,k}_{r_h} + P^n_{r_h}\psi_0 + O^n_{r_h})(x)\mathrm{d}s \\
&= v_t(x) - v^k_t(x) - P_{t-s}v_s(x) + P_{t-s}v^k_s(x) \\
&\quad + \int_s^t \Big( P_{(t-r)} b^k(v_r + P_r\psi_0 + O_r)(x) \\
&\quad - \int_s^t P^n_{(t-r)_h} b^k(v^{h,k}_{r_h} + P^n_{r_h}\psi_0 + O^n_{r_h}(x))(x)\Big)\mathrm{d}s \\
&:= V^k_t - P_{t-s}V^k_s + \mathcal{E}^{1,h,k}_{s,t} + \mathcal{E}^{2,h,k}_{s,t} + \mathcal{E}^{3,h,k}_{s,t}, \\
&:= \epsilon(h,k) + \mathcal{E}^{1,h,k}_{s,t}
\end{aligned}$$

# Elements of Proof II

$$\mathcal{E}^{1,h,k}_{s,t} = \int_0^t \int_0^1 p_{t-r}(x,y)\Big(b^k(v_r(y) + P_r\psi_0(y) + O_r(y)) - b^k(v^{h,k}_r(y) + P^n_r\psi_0(y) + O^n_r(y))\Big)\mathrm{d}y\mathrm{d}r$$

$$\mathcal{E}^{2,h,k}_{s,t} = \int_0^t \int_0^1 p_{t-r}(x,y)\Big(b^k(v^{h,k}_r(y) + P^n_r\psi_0(y) + O^n_r(y)) - b^k(v^{h,k}_{r_h}(y_n) + P^n_{r_h}\psi_0(y_n) + O^n_{r_h}(y_n))\Big)\mathrm{d}y\mathrm{d}r$$

$$\mathcal{E}^{3,h,k}_{s,t} = \int_0^t \int_0^1 (p_{t-r} - p^n_{(t-r)_h})(x,y) b^k(v^{h,k}_{r_h}(y_n) + P^n_{r_h}\psi_0(y_n) + O^n_{r_h}(y_n))\mathrm{d}y\mathrm{d}r.$$

will be controlled on small intervals.

# Elements of Proof III

On all intervals $[S, T]$, we have

$$[V^k]_{\mathcal{C}^{\frac{1}{2}}_{[S,T]} L^m} \leq C \|b - b^k\|_{\mathcal{B}^{\gamma-1}_p}.$$

$$[\mathcal{E}^{2,h,k}]_{\mathcal{C}^{\frac{1}{2}}_{[S,T]} L^m} + [\mathcal{E}^{3,h,k}]_{\mathcal{C}^{\frac{1}{2}}_{[S,T]} L^m} \leq C\Big( (1 + \|b^k\|_\infty) n^{-\frac{1}{2}+\varepsilon} + (1 + \|b^k\|_\infty)(1 + \|b^k\|_{\mathcal{C}^1}) n^{-1+\varepsilon} \Big).$$

# Elements of Proof III

On all intervals $[S, T]$, we have

$$[V^k]_{\mathcal{C}^{\frac{1}{2}}_{[S,T]}L^m} \leq C\|b - b^k\|_{\mathcal{B}_p^{\gamma-1}}.$$

$$[\mathcal{E}^{2,h,k}]_{\mathcal{C}^{\frac{1}{2}}_{[S,T]}L^m} + [\mathcal{E}^{3,h,k}]_{\mathcal{C}^{\frac{1}{2}}_{[S,T]}L^m} \leq C\Big((1 + \|b^k\|_\infty)n^{-\frac{1}{2}+\varepsilon} + (1 + \|b^k\|_\infty)(1 + \|b^k\|_{\mathcal{C}^1})n^{-1+\varepsilon}\Big).$$

which reads

$$\epsilon(h,k) \leq C\Big(\|b - b^k\|_{\mathcal{B}_p^{\gamma-1}} + (1 + \|b^k\|_\infty)n^{-\frac{1}{2}+\varepsilon} + (1 + \|b^k\|_\infty)(1 + \|b^k\|_{\mathcal{C}^1})n^{-1+\varepsilon}\Big).$$

# Elements of Proof III

On all intervals $[S, T]$, we have

$$[V^k]_{\mathcal{C}^{\frac{1}{2}}_{[S,T]}L^m} \leq C\|b - b^k\|_{\mathcal{B}_p^{\gamma-1}}.$$

$$[\mathcal{E}^{2,h,k}]_{\mathcal{C}^{\frac{1}{2}}_{[S,T]}L^m} + [\mathcal{E}^{3,h,k}]_{\mathcal{C}^{\frac{1}{2}}_{[S,T]}L^m} \leq C\Big((1 + \|b^k\|_\infty)n^{-\frac{1}{2}+\varepsilon} + (1 + \|b^k\|_\infty)(1 + \|b^k\|_{\mathcal{C}^1})n^{-1+\varepsilon}\Big).$$

which reads

$$\epsilon(h,k) \leq C\Big(\|b - b^k\|_{\mathcal{B}_p^{\gamma-1}} + (1 + \|b^k\|_\infty)n^{-\frac{1}{2}+\varepsilon} + (1 + \|b^k\|_\infty)(1 + \|b^k\|_{\mathcal{C}^1})n^{-1+\varepsilon}\Big).$$

The last step is to obtain

$$[\mathcal{E}^{1,h,k}]_{\mathcal{C}^{\frac{1}{2}}_{[S,T]}L^m} \leq C\Big([\mathcal{E}^{h,k}]_{\mathcal{C}^{\frac{1}{2}}_{[S,T]}L^m} + [\mathcal{E}^{h,k}]_{\mathcal{C}^{\frac{1}{2}}_{[0,S]}L^m} + \epsilon(h,k)\Big)(T - S)^{\frac{1}{4}(\gamma+1-1/p)}.$$

# Elements of Proof IV

Hence the bound

$$[\mathcal{E}^{h,k}]_{\mathcal{C}^{\frac{1}{2}}_{[S,T]}L^m} \leq \epsilon(h,k) + C\Big([\mathcal{E}^{h,k}]_{\mathcal{C}^{\frac{1}{2}}_{[S,T]}L^m} + [\mathcal{E}^{h,k}]_{\mathcal{C}^{\frac{1}{2}}_{[0,S]}L^m}\Big)(T-S)^{\frac{1}{4}(\gamma+1-1/p)}.$$

leads to

$$[\mathcal{E}^{h,k}]_{\mathcal{C}^{\frac{1}{2}}_{[0,1]}L^m} \leq C\epsilon(h,k),$$

which is controlled.

# Theoretical Results

Let $\gamma \in \mathbb{R}$, $p \in [1, \infty]$ such that

$$0 > \gamma - \frac{1}{p} \geq -1 \text{ and } \gamma > -1. \tag{A1}$$

Let $b \in \mathcal{B}_p^\gamma$, $m \in [2, \infty)$, $\varepsilon \in (0, 1/2)$ and let $\psi_0 \in \mathcal{C}^{\frac{1}{2}-\varepsilon}([0,1], \mathbb{R})$.

Let $u$ be the strong solution to the SPDE with drift $b$.

Let $(b^k)$ be a sequence of smooth functions that converges to $b$ in $\mathcal{B}_p^{\gamma-}$ and $(u^{h,k})_{h\in(0,1),k\in\mathbb{N}}$ be the tamed Euler finite-differences scheme defined on the same probability space and with the same space-time white noise $\xi$ as $u$.

# Theoretical Results

**Theorem (G.-Haress-Richard)**

*(a) Regularity of the tamed Euler scheme: Let* $\eta \in \left(0, \frac{1}{2}\right)$, $\mathcal{D}$ *be a subset of* $[0,1] \times \mathbb{N}$ *and assume that*

$$\sup_{(h,k)\in\mathcal{D}} \|b^k\|_\infty h^{\frac{1}{4}-\eta} < \infty \quad \text{and} \quad \sup_{(h,k)\in\mathcal{D}} \|b^k\|_{\mathcal{C}^1} h^{\frac{1}{2}} < \infty, \tag{A2}$$

*then* $\sup_{(h,k)\in\mathcal{D}} \{u^{h,k} - O\}_{\mathcal{C}^{\frac{1}{2}+\eta}_{[0,1]} L^{m,\infty}} < \infty.$

# Theoretical Results

**Theorem (G.-Haress-Richard)**

*Assume that* $[u - O]_{\mathcal{C}^{3/4}_{[0,1]} L^{m,\infty}} < \infty$.

*(b) The sub-critical case: If* $-1 < \gamma - d/p < 0$*, then there exists a constant* $C$ *that depends on* $m, p, \gamma, \varepsilon, \|b\|_{\mathcal{B}^{\gamma}_{p}}, \|\psi_0\|_{\mathcal{C}^{\frac{1}{2}-\varepsilon}}$ *such that for all* $h \in (0,1)$ *and* $k \in \mathbb{N}$*, the following bound holds:*

$$[\mathcal{E}^{h,k}]_{\mathcal{C}^{\frac{1}{2}}_{[0,1]} L^m} \leq C \left( \|b^k - b\|_{\mathcal{B}^{\gamma-1}_{p}} + (1 + \|b^k\|_\infty) n^{-\frac{1}{2}+\varepsilon} + \|b^k\|_\infty \|b^k\|_{\mathcal{C}^1} n^{-1+\varepsilon} \right).$$

# Theoretical Results

**Theorem (G.-Haress-Richard)**

*Assume that* $[u-O]_{\mathcal{C}^{3/4}_{[0,1]}L^{m,\infty}} < \infty$.

*(c) The critical case: If* $\gamma - 1/p = -1$ *and* $p < +\infty$. *If* (A2) *holds, then there exists two constants* $C_1, C_2$ *that depend on* $m, p, \gamma, \varepsilon, \|b\|_{\mathcal{B}^{\gamma}_p}, \|\psi_0\|_{\mathcal{C}^{\frac{1}{2}-\varepsilon}}$ *such that for all* $h \in (0,1)$ *and* $k \in \mathbb{N}$, *the following bound holds:*

$$[\mathcal{E}^{h,k}_{0,\cdot}]_{L^{\infty}_{[0,1]}L^m} \leq C_1\Big(\|b-b^k\|_{\mathcal{B}^{\gamma-1}_p}(1+|\log\|b-b^k\|_{\mathcal{B}^{\gamma-1}_p}|)+(1+\|b^k\|_{\infty})n^{-\frac{1}{2}+\varepsilon} + (1+\|b^k\|_{\infty})(1+\|b^k\|_{\mathcal{C}^1})n^{-1+\varepsilon}\Big)^{C_2}.$$

# Theoretical Results

| | | | |
|---|---|---|---|
| *Regularity* | $\gamma - \frac{1}{p} = 0$ | $\gamma - \frac{1}{p} \in (-1, 0)$ | $\gamma - \frac{1}{p} = -1$ and $p < \infty$ |
| *Space Order* | $\frac{1}{2} - \varepsilon$ | $\frac{1}{2 - 2(\gamma - \frac{1}{p})} - \varepsilon$ | $C > 0$ |
| *Time Order* | $\frac{1}{4} - \varepsilon$ | $\frac{1}{4 - 4(\gamma - \frac{1}{p})} - \varepsilon$ | $C/2 > 0$ |

Table: Rate of convergence of the tamed Euler finite-differences scheme depending on the Besov regularity of the drift.

# Numerical simulation - Stochastic heat equation with Dirac drift

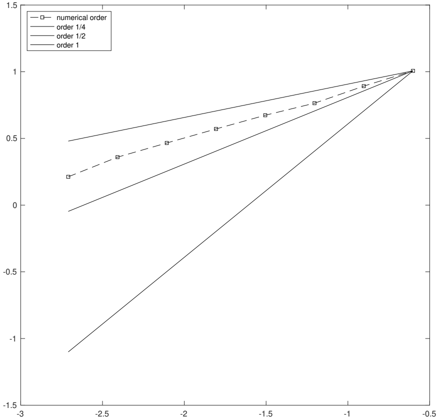
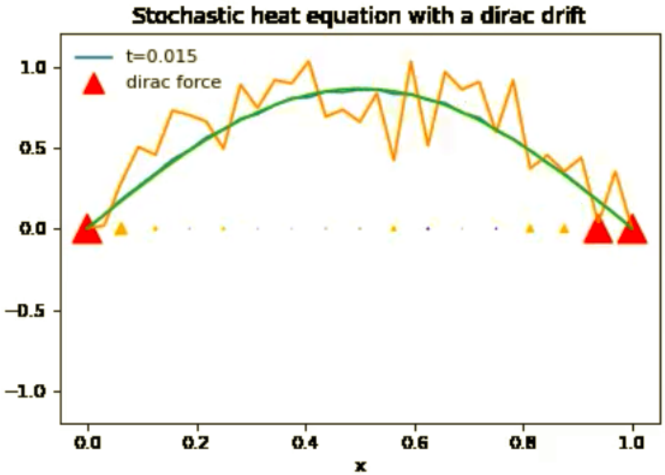
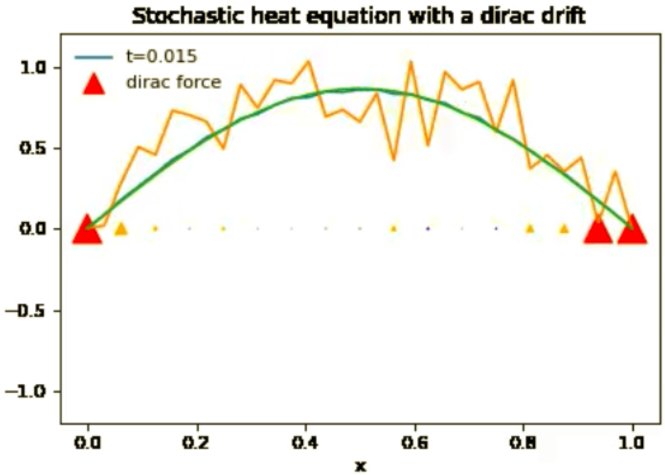



Figure: Approximate slope is 0.36

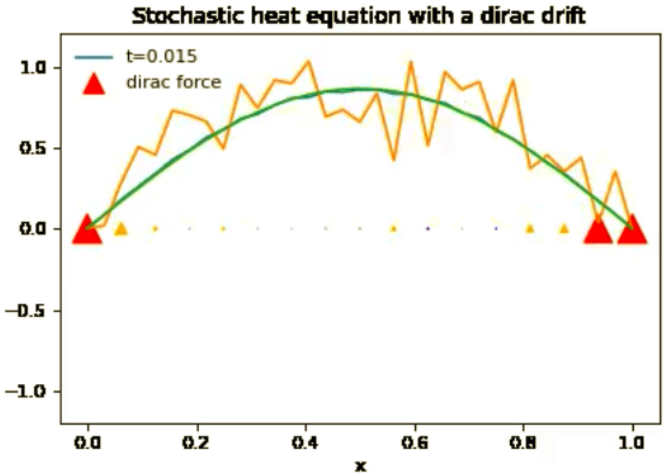
# Numerical simulation - regularized Dirac in 0



## Numerical simulation - Dirac in 0



# Numerical simulation - Dirac in 1



Thanks for your attention !

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