Stochastics Models of Interfaces with Damage: A Numerical Study

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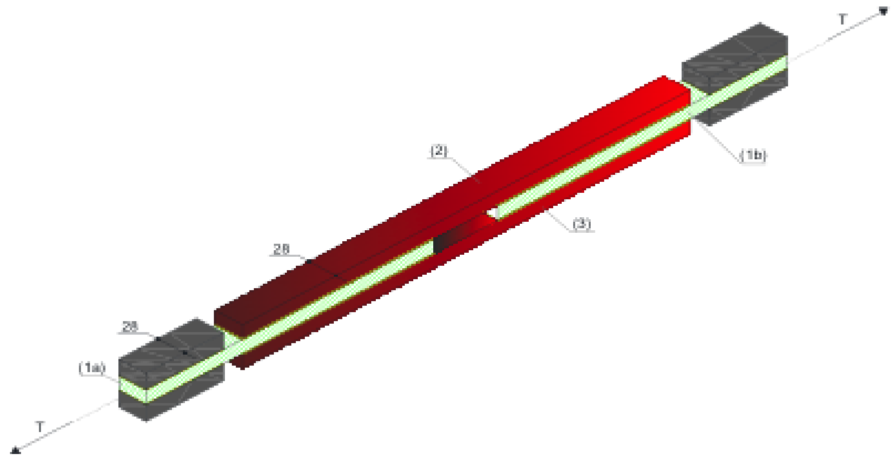
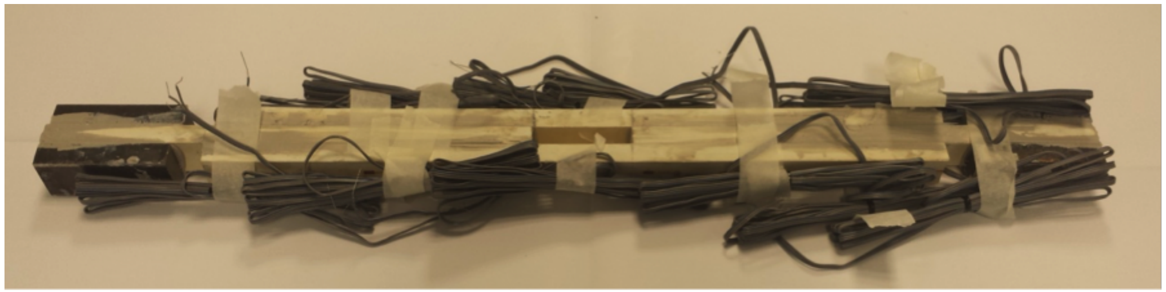
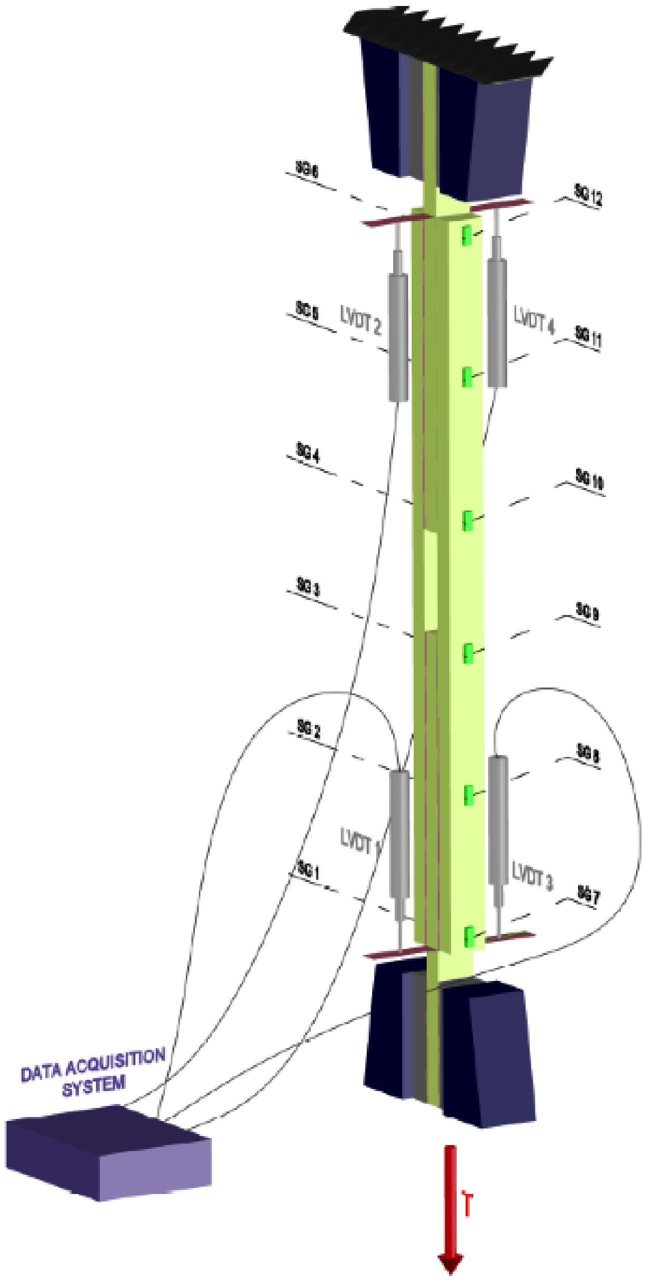
Application: crack propagation, exemple provided by [OMDL16]





- Adherents: GFRP;
- Glue: epoxy resin.

Joint works with: C. Bauzet, F. Nabet, F. Lebon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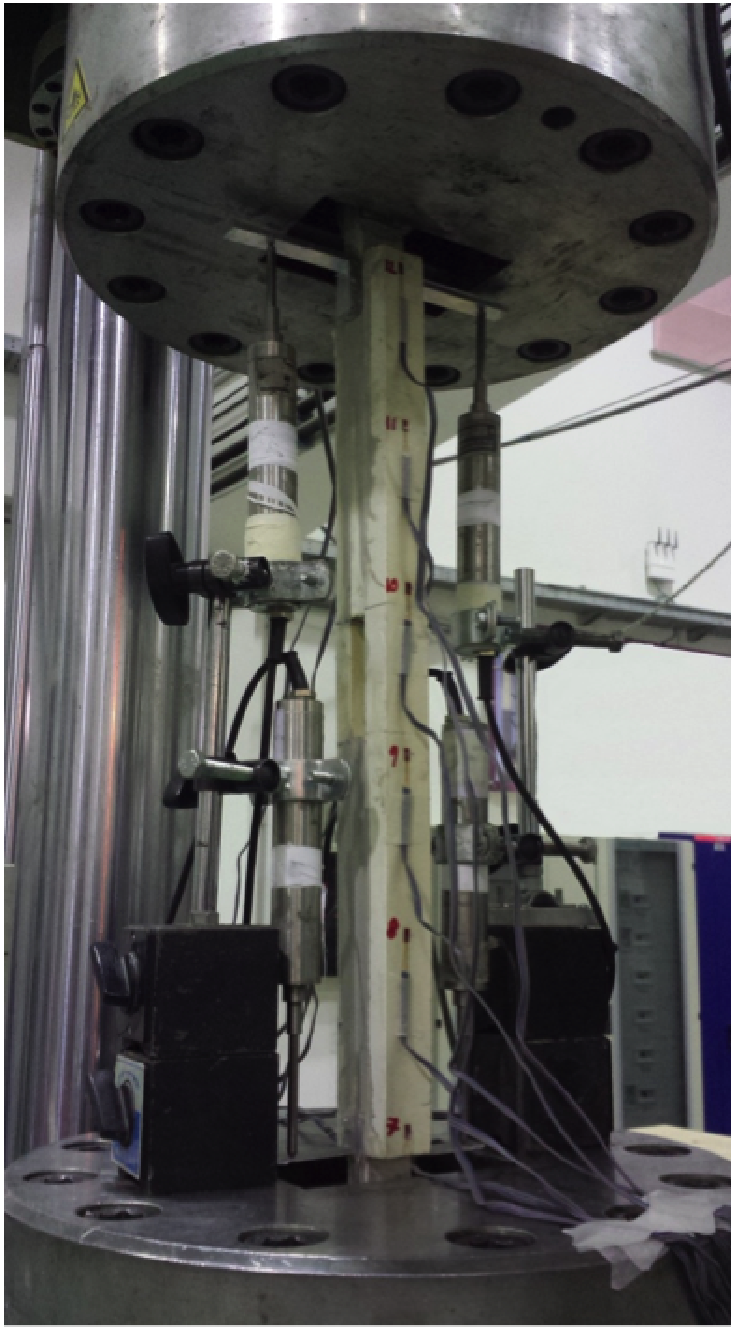
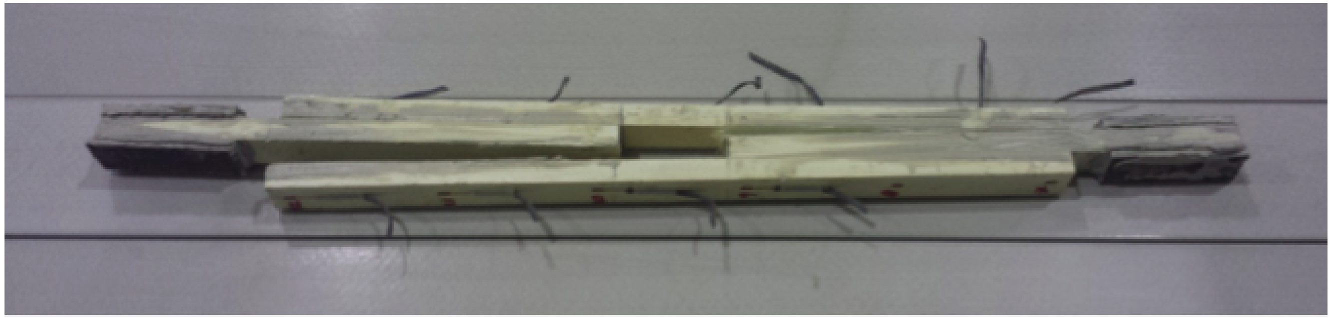
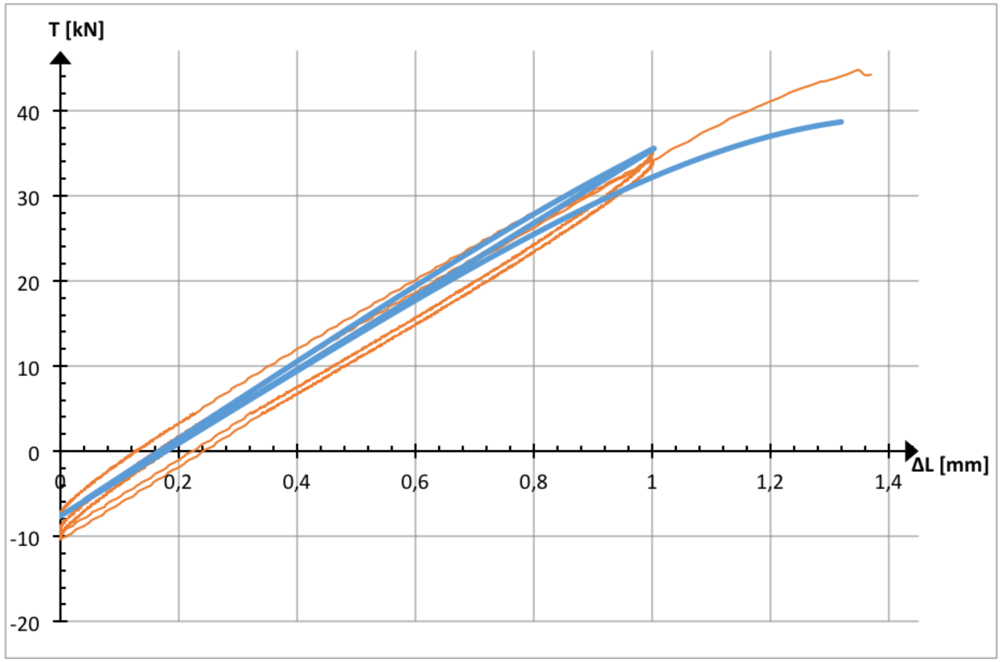


Results





Objectives of the study

Numerical simulations of adhesive behaviour:

- with interface approximating law;
- with damage;
- with stochastic effects.

Plan of the talk

A General Model of Damaging Materials

For this part, we follow [**LRR23**]. For that purpose, we define

- the displacement field u;
- the elastic strain tensor $e(u)$ which is the symmetrical gradient of u;
- A variable R which the a crack density and represents an internal variable of damage.

A specific free energy potential is chosen as

$$\psi(e(u), R) = \frac{1}{2}\mathbb{K}(R)e(u) : e(u) + \omega(R) + \frac{\alpha}{p}|\nabla R|^p + \chi_{[0,1]}(R) \tag{3.1}$$

where

- $\mathbb{K}(R)$ is the stiffness tensor of the material (adhesive);
- χ_A is the indicator function of the set A:

$$\chi_A(x) = \begin{cases} 0 & \text{if } x \in A \\ +\infty & \text{if } x \notin A \end{cases}$$

- the reals p and α are materials parameters;
- $\omega(R)$ is an activation energy of damage;
- $|\nabla R|^p$ models the non-local character of damage.

Evolution of the state variable R

For that, we introduce a dissipation potential Φ:

$$\Phi(\dot{R}) = \frac{1}{\beta+1}\eta(R)\dot{R}^{\beta+1} + \chi_{[0;+\infty[}(\dot{R})$$

where

- β (which controls the damage velocity) and $\eta > 0$ (viscosity parameter) are material parameters;
- $\chi_{[0;+\infty[}(\dot{R})$ means that the damage is irreversible.

Then, the evolution law of the damage R is

$$\begin{cases} \eta(R)\dot{R}^{\beta} = -\left(\omega_{,R}(R) + \frac{1}{2}\mathbb{K}_{,R}(R)e(u)e(u) + \alpha\Delta_p R\right)_- \\ R(0) = R_0 \end{cases} \tag{3.2}$$

where $(\cdot)_-$ denotes the negative part of a function and $\Delta_p R$ is the p-laplacian of R.

Exemple in a 1D case

In this case, the evolution of the damage R, equation (3.2), becomes:

$$\eta(R)\dot{R}^{\beta} = -\left(\omega_{,R}(R) + \frac{1}{2}E_{,R}(R)\epsilon^2\right)_-$$

where

- ϵ is the uniaxial strain;
- $E_{,R}(R)$ represents the derivative w.r.t. R of the Young's modulus of the damaged adhesive.

Using **[WG10]**, one has

$$E(R) = E_0(1 - 2\pi R)$$

where E_0 is the Young's modulus of the undamaged material.

For the numerical simulations, a linear strain ramp is imposed: $\epsilon(t) = \dot{\epsilon}t$ the equation (3.2) becomes

$$\dot{R} = \frac{1}{\eta(R)}\left[-\left(\omega_{,R}(R) - \pi E_0\dot{\epsilon}^2 t^2\right)_-\right]^{\frac{1}{\beta}} \qquad (3.3)$$

Data

- $\omega_{,R}(R) \equiv \bar{\omega} = 0.06$ Pa;
- $\eta(R) \equiv \bar{\eta} = 3.6\ 10^2$ Pa;
- $R_0 = 0$ (undamaged adhesive at the beginning).

We also consider the normalised time $t = \frac{1}{\dot{\epsilon}} \left(\frac{\bar{\omega}}{\pi E_0} \right)^{\frac{1}{2}} \frac{1}{2.7}$.

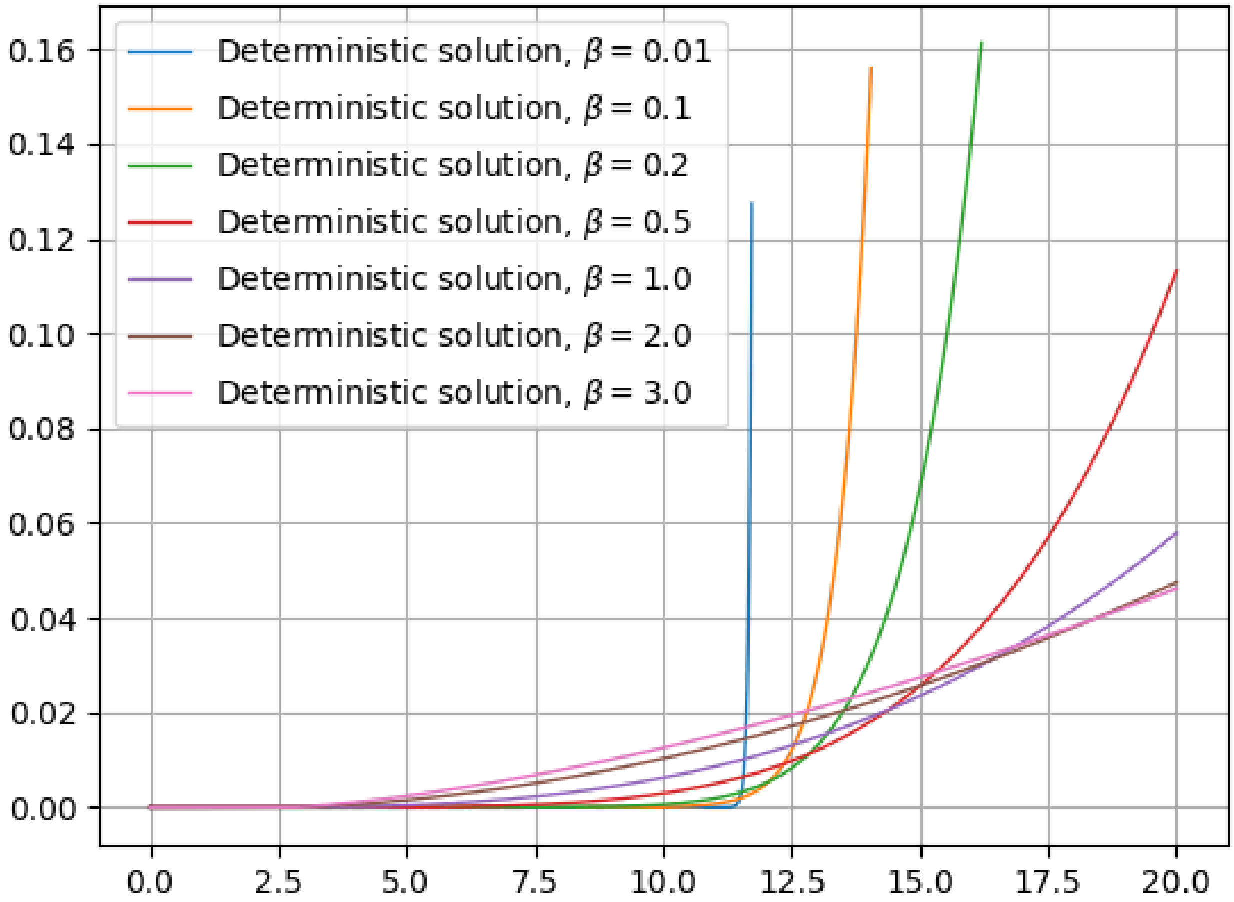


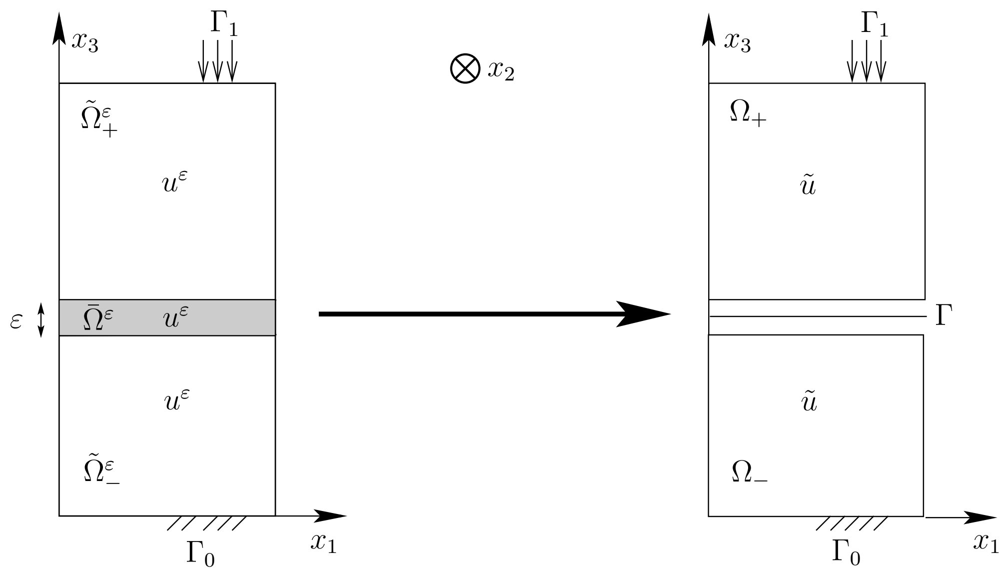
Figure 1: Evolution of the damage variable w.r.t. time for various values of β

Modeling of interfaces

Issue

To a given set of elastics bodies glued together with an interphase with a small thickness,

what is the good modeling to correctly approximate the behavior of the interphase by an interface law ?



Governing equilibrium equations

$$
\left\{
\begin{array}{lll}
\mathrm{div}\sigma^\varepsilon + f = 0 & \text{in } \Omega^\varepsilon_\pm \cup \Omega^\varepsilon & \\
\sigma^\varepsilon n = g & \text{on } \Gamma_1 & \\
u^\varepsilon = u_d & \text{on } \Gamma_0 & (4.1)\\
\sigma^\varepsilon = \bar{\mathbb{K}}^\varepsilon_\pm e(u^\varepsilon) & \text{in } \Omega^\varepsilon_\pm & \\
\sigma^\varepsilon = \hat{\mathbb{K}}^\varepsilon e(u^\varepsilon) & \text{in } \Omega^\varepsilon &
\end{array}
\right.
$$

where $e(u^\varepsilon) = \nabla_{sym} u^\varepsilon$.

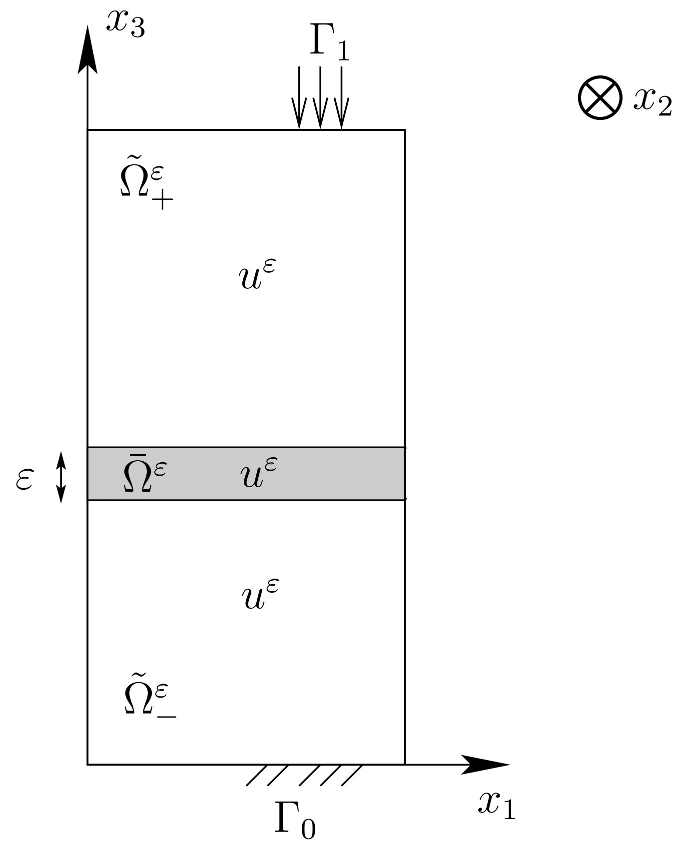


Figure 2: Geometry of the initial problem

Variational formulation

$$\left\{\begin{array}{l}\text{Find } u^\varepsilon \in V(\Omega^\varepsilon) \text{ such that} \\ \bar{A}_-^\varepsilon(u^\varepsilon, v^\varepsilon) + \bar{A}_+^\varepsilon(u^\varepsilon, v^\varepsilon) + \hat{A}^\varepsilon(u^\varepsilon, v^\varepsilon) = L^\varepsilon(v^\varepsilon),\end{array}\right. \tag{4.2}$$

for all $u^\varepsilon \in V(\Omega^\varepsilon)$, where

- The functional space: $V(\Omega^\varepsilon) := \{u^\varepsilon \in H^1(\Omega^\varepsilon; \mathbb{R}^3);\ u^\varepsilon = \mathbf{0} \text{ on } \Gamma_u^\varepsilon\}$,
- the bilinear forms in the adherents are

$$\bar{A}_\pm^\varepsilon(u^\varepsilon, v^\varepsilon) := \int_{\Omega_\pm^\varepsilon} \bar{\mathbb{K}}^\varepsilon \boldsymbol{\nabla}^\varepsilon u^\varepsilon \cdot \boldsymbol{\nabla}^\varepsilon v^\varepsilon dx^\varepsilon \tag{4.3}$$

- the bilinear form in the adhesive is

$$\hat{A}^\varepsilon(u^\varepsilon, v^\varepsilon) := \int_{\Omega^\varepsilon} \hat{\mathbb{K}}^\varepsilon \boldsymbol{\nabla}^\varepsilon u^\varepsilon \cdot \boldsymbol{\nabla}^\varepsilon v^\varepsilon dx^\varepsilon \tag{4.4}$$

- and the linear form $L^\varepsilon(\cdot)$ is defined by

$$L^\varepsilon(v^\varepsilon) := \int_{\Omega^\varepsilon_\pm} \mathbf{f} \cdot v^\varepsilon dx^\varepsilon + \int_{\Gamma^\varepsilon_g} \mathbf{f} \cdot v^\varepsilon d\Gamma^\varepsilon.$$

By virtue of the regularity of the loads, the positivity of the constitutive matrices and thanks to the Lax-Milgram's lemma, problem admits one and only one solution.

Now, in the equation (4.2):

$$\begin{cases} \text{Find } u^\varepsilon \in V(\Omega^\varepsilon) \text{ such that} \\ \bar{A}^\varepsilon_-(u^\varepsilon, v^\varepsilon) + \bar{A}^\varepsilon_+(u^\varepsilon, v^\varepsilon) + \hat{A}^\varepsilon(u^\varepsilon, v^\varepsilon) = L^\varepsilon(v^\varepsilon), \end{cases}$$

we want to approximate $\hat{A}^\varepsilon$ by an integral on the surface Γ (interface condition).

Main ideas of the method

Computation of the interface law: make asymptotic expansions in term of the small parameter ε

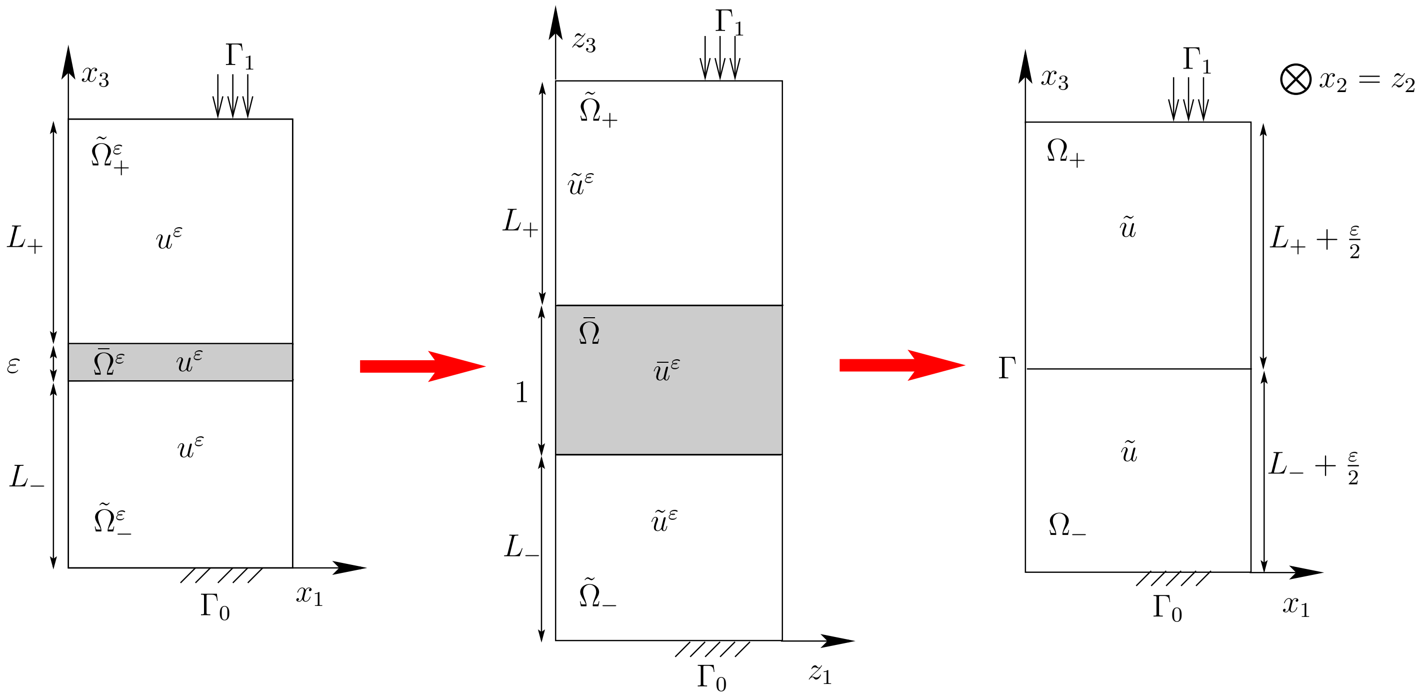


Figure 3: The change of variable

Changes of variables: (like in homogenization methods)

- dilatation in the interphase (adhesive): $(z_1, z_2, z_3) = (x_1, x_2, \frac{x_3}{\varepsilon}) \Longrightarrow \frac{\partial}{\partial z_3} = \frac{1}{\varepsilon}\frac{\partial}{\partial x_3}$;
- translations in the adherents: $(z_1, z_2, z_3) = (x_1, x_2, x_3 \pm \frac{1-\varepsilon}{2})$.

Assumptions on constitutive matrices:

We assume that the constitutive matrices in $\Omega_{\pm}^{\varepsilon}$ are independent of ε,

$$\bar{\mathbb{K}}^{\varepsilon} = \bar{\mathbb{K}},$$

while the constitutive coefficients of Ω^{ε} present the following dependences on ε:

$$\hat{\mathbb{K}}^{\varepsilon} = \varepsilon^{p} \hat{\mathbb{K}},$$

with $p \in \{-1, 0, 1\}$.

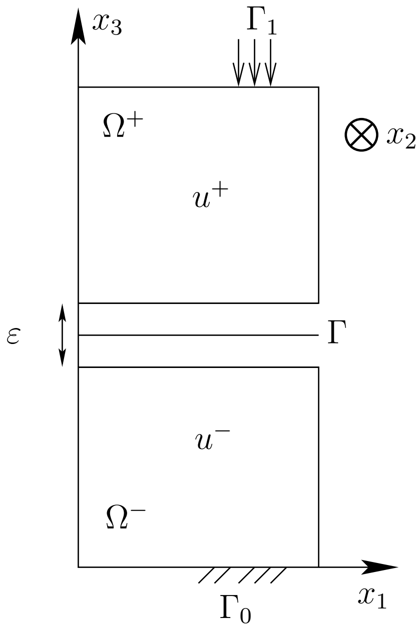
Three different limit behaviors will be characterized according to the choice of the exponent p:

- in the case of $p = -1$, we derive a model for a *rigid* interface (reinforcement, welding);
- in the case of $p = 0$, we derive a model for a *hard* interface;
- in the case of $p = 1$, we deduce a model for a *soft* interface.

Inside the interphase, we seek the solution as an asymptotic expansion with respect to ε:

$$\begin{cases} u^{\varepsilon} = \hat{u}^0 + \varepsilon \hat{u}^1 + \varepsilon^2 \hat{u}^2 + \ldots & \text{(Displacement)} \\ \sigma^{\varepsilon} = \hat{\sigma}^0 + \varepsilon \hat{\sigma}^1 + \varepsilon^2 \hat{\sigma}^2 + \ldots & \text{(Stress)} \end{cases} \tag{4.5}$$

Interface conditions: notations



- Interface colinear to (x_1, x_2): $\Gamma = \{(x_1, x_2, x_3) \in \Omega,\ x_3 = C_0\}$;
- Jump across the interface:
  $[f](x_1, x_2) = \lim_{x_3 \to C_0 + \frac{\varepsilon}{2}} f(x_1, x_2, x_3) - \lim_{x_3 \to C_0 - \frac{\varepsilon}{2}} f(x_1, x_2, x_3)$
- Average on the interface:
  $< f >(x_1, x_2) = \frac{1}{2} \left(\lim_{x_3 \to C_0 + \frac{\varepsilon}{2}} f(x_1, x_2, x_3) + \lim_{x_3 \to C_0 - \frac{\varepsilon}{2}} f(x_1, x_2, x_3) \right)$

Interface conditions in the case of soft interface ($p = 1$)

Using the rescaling and identifying at each order, one obtain the interface conditions

- Order 0

Governing equations

$$\begin{cases} -\text{div } \bar{\sigma}^0 = f & \text{in } \Omega_\pm, \\ \bar{\sigma}^0 n = g & \text{on } \Gamma_1, \\ \bar{u}^0 = 0 & \text{on } \Gamma_0, \end{cases}$$

Transmission conditions on $\Gamma_\pm$

$$\begin{cases} [\bar{u}^0] = (\hat{\mathbb{K}}_{33})^{-1} \langle \bar{\sigma}^0 e_3 \rangle, \\ [\bar{\sigma}^0 e_3] = 0. \end{cases} \tag{4.6}$$

- Order 1

Governing equations

$$\begin{cases} -\text{div } \bar{\sigma}^1 = 0 & \text{in } \Omega_\pm, \\ \bar{\sigma}^1 n = 0 & \text{on } \Gamma_1, \\ \bar{u}^1 = 0 & \text{on } \Gamma_0, \end{cases}$$

Transmission conditions on $\Gamma_\pm$

$$\begin{cases} [\bar{u}^1] = (\hat{\mathbb{K}}_{33})^{-1} \left(\langle \bar{\sigma}^1 e_3 \rangle - \hat{\mathbb{K}}_{\alpha 3} \langle \bar{u}^0 \rangle_{,\alpha} \right), \\ [\bar{\sigma}^1 e_3] = -\hat{\mathbb{K}}_{3\alpha} [\bar{u}^0]_{,\alpha}. \end{cases} \tag{4.7}$$

Interface conditions in the case of hard interface ($p = 0$)

One then obtain the interface conditions

- Order 0

Governing equations

$$\begin{cases} -\text{div } \bar{\sigma}^0 = f & \text{in } \Omega_\pm, \\ \bar{\sigma}^0 n = g & \text{on } \Gamma_1, \\ \bar{u}^0 = 0 & \text{on } \Gamma_0, \end{cases}$$

Transmission conditions on $\Gamma_\pm$

$$\begin{cases} [\bar{u}^0] = 0, \\ [\bar{\sigma}^0 e_3] = 0. \end{cases} \quad (4.8)$$

- Order 1

Governing equations

$$\begin{cases} -\text{div } \bar{\sigma}^1 = 0 & \text{in } \Omega_\pm, \\ \bar{\sigma}^1 n = 0 & \text{on } \Gamma_1, \\ \bar{u}^1 = 0 & \text{on } \Gamma_0, \end{cases}$$

Transmission conditions on $\Gamma_\pm$

$$\begin{cases} [\bar{u}^1] = (\hat{\mathbb{K}}_{33})^{-1} \left(\langle \bar{\sigma}^0 e_3 \rangle - \hat{\mathbb{K}}_{\alpha 3} \langle \bar{u}^0 \rangle_{,\alpha} \right), \\ [\bar{\sigma}^1 e_3] = - \left(\hat{\mathbb{K}}_{3\alpha} [\bar{u}^1]_{,\alpha} + \hat{\mathbb{K}}_{\alpha\beta} \langle \bar{u}^0 \rangle_{,\alpha\beta} \right). \end{cases} \quad (4.9)$$

Implicit unified interface conditions

We denote by

- $\tilde{u}^\varepsilon := \bar{u}^0 + \varepsilon \bar{u}^1$;
- $\tilde{\sigma}^\varepsilon := \bar{\sigma}^0 + \varepsilon \bar{\sigma}^1$

two suitable approximations for $\bar{u}^\varepsilon$ and $\bar{\sigma}^\varepsilon$.

An alternative expression of the above transmission conditions can be given in terms of $\langle \tilde{\sigma}^\varepsilon e_3 \rangle$ and $[\tilde{\sigma}^\varepsilon e_3]$, which will be useful to write the variational formulation of the interface problem:

$$\begin{cases} \langle \tilde{\sigma}^\varepsilon e_3 \rangle = \frac{1}{\varepsilon} \hat{\mathbb{K}}^\varepsilon_{33} [\tilde{u}^\varepsilon] + \hat{\mathbb{K}}^\varepsilon_{\alpha 3} \langle \tilde{u}^\varepsilon \rangle_{,\alpha} + o(\varepsilon^2), \\ [\tilde{\sigma}^\varepsilon e_3] = -\hat{\mathbb{K}}^\varepsilon_{3\alpha} [\tilde{u}^\varepsilon]_{,\alpha} - \varepsilon \hat{\mathbb{K}}^\varepsilon_{\alpha\beta} \langle \tilde{u}^\varepsilon \rangle_{,\alpha\beta} + o(\varepsilon^2). \end{cases} \tag{4.10}$$

Remarks

- In the following, we will consider a soft interface law at order 0;
- With this approximation, the displacement u is approximated by a linear function in the third direction, then the strain is considered as constant inside the interphase and the damage evolution equation becomes

$$\eta(R)\dot{R}^{\beta} = -\Big(\omega_{,R}(R) + \frac{1}{2}K^{33}_{,R}(R)[u][u] + \alpha\Delta_p^2 R\Big)_{-} \tag{4.11}$$

Introduction of Stochastic Effects

We start again from the article of [**LRR23**].

We remains that the equation of the evolution of R is

$$\begin{cases} \eta(R)\dot{R}^{\beta} = -\left(\omega_{,R}(R) + \frac{1}{2}K_{,R}(R)e(u)e(u) + \alpha\Delta_p R\right)_{-} \\ R(0) = R_0 \end{cases} \tag{5.1}$$

which becomes for the 1D case with some additional hypothesis on the loading:

$$\begin{cases} \dot{R} = \frac{1}{\eta(R)}\left[-\left(\omega_{,R}(R) - \pi E_0 \dot{\varepsilon}^2 t^2\right)_{-}\right]^{\frac{1}{\beta}} \\ R(0) = R_0 \end{cases} \tag{5.2}$$

We propose to introduce some stochastic effects by using the following Stochastic Ordinary Differential Equation:

$$\begin{cases} \partial_t \left(R + \int_0^t h(R) dW \right) = \frac{1}{\eta(R)} \left[- \left(\omega_{,R}(R) - \pi E_0 \dot{\varepsilon}^2 t^2 \right)_- \right]^{\frac{1}{\beta}} \\ R(0) = R_0 \end{cases} \tag{5.3}$$

for some given function h.

Writing

$$f(t,R) = \frac{1}{\eta(R)} \left[- \left(\omega_{,R}(R) - \pi E_0 \dot{\varepsilon}^2 t^2 \right)_- \right]^{\frac{1}{\beta}},$$

(5.3) can be written

$$dR + h(R) dW = f(t,R) dt \tag{5.4}$$

Numerical Algorithms

In order to approximate the solutions of problem (5.4), let us introduce a time step δt and a discrete sequence of time $t_n = n\delta t$, $n \in \mathbb{N}$,

Then, 2 classical numerical schemes can be introduced:

1. The Euler-Maruyama method [**KP92**]

$$R_{n+1} \simeq R_n + f(t_n, R_n)\delta t - h(R_n)dW_{n+1} \tag{5.5}$$

   which has an order of weak convergence equal to 1 and an order of strong convergence equal to $\frac{1}{2}$;

2. The Milstein method [**M75**]

$$R_{n+1} \simeq R_n + f(t_n, R_n)\delta t - h(R_n)dW_{n+1} + \frac{1}{2}h(R_n)h'(R_n)(dW_{n+1}^2 - \delta t) \tag{5.6}$$

   which has an order of weak convergence equal to 1 and an order of strong convergence equal to 1.

Numerical Exemples

Comparison between the deterministic and the 2 stochastic methods ($h(w) = 10^{-1}$, constant).

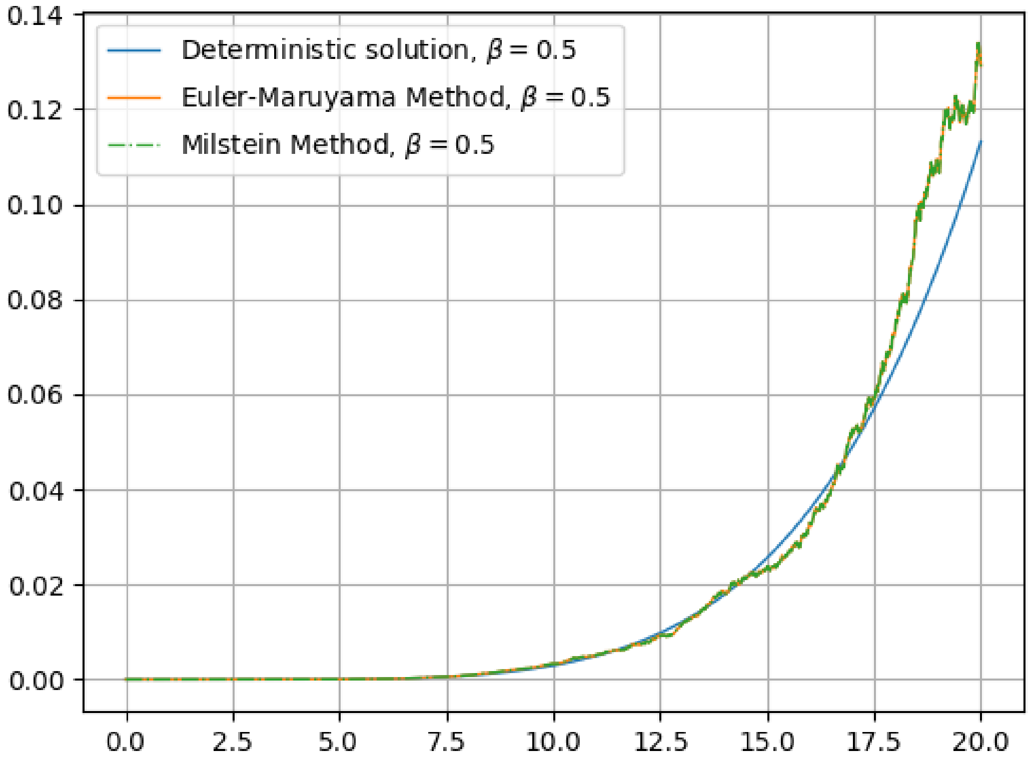


Figure 4: Comparizon deterministic/stochastic

Other data: $\delta t = 10^{-2} s$, $\beta = 0.5$, $\omega_{,R}(R) = \bar{\omega} = 0.06$ Pa; $\eta(R) + \bar{\eta} = 3.6\ 10^2$ Pa; $R_0 = 0$ (undamaged adhesive at the beginning).

Remark on the increasing of the damage

In the previous modeling, the increasing of the damage is not ensured.
In order to ensure the irreversibility of the damage, the previous algorithm can be adapted:

1. The Euler-Maruyama method

$$R_{n+1} \simeq R_n + \text{Max}\Big(0; f(t_n, R_n)\delta t - h(R_n)dW_{n+1}\Big) \tag{5.7}$$

2. The Milstein method

$$R_{n+1} \simeq R_n + \text{Max}\Big(0; f(t_n, R_n)\delta t - h(R_n)dW_{n+1} + \frac{1}{2}h(R_n)h'(R_n)(dW_{n+1}^2 - \delta t)\Big) \tag{5.8}$$

Comparizon deterministic solution and Stochastic simulations with a growing constraint

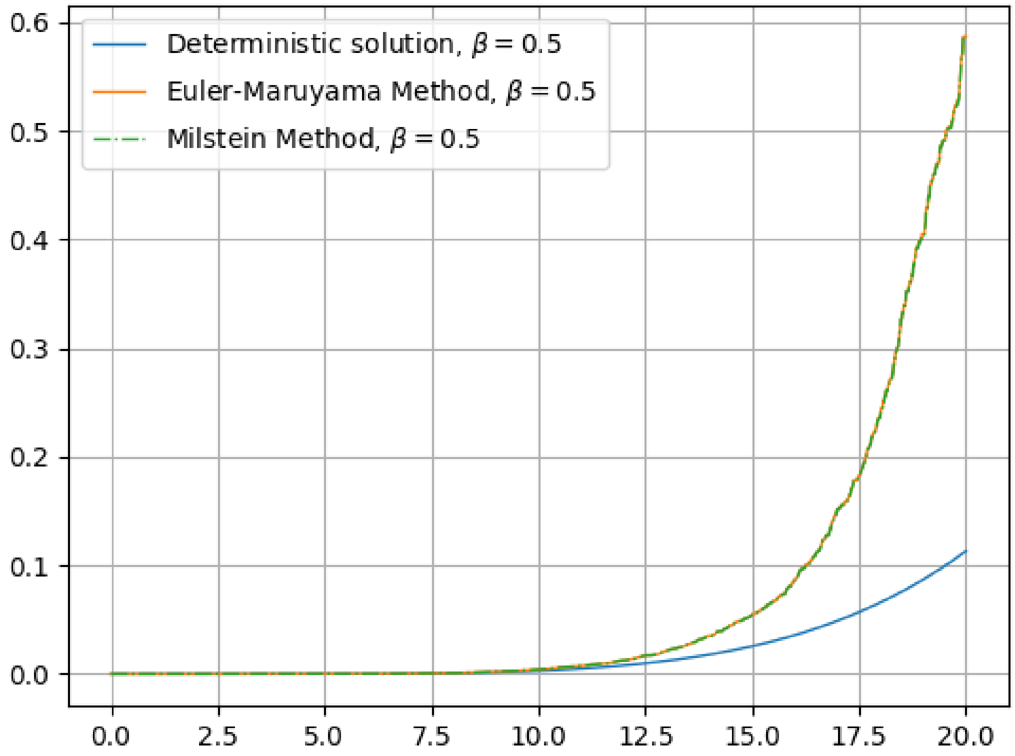


Figure 5: Comparizon deterministic solution and Stochastic simulations with a growing constraint

Estimation of the Expectation

We estimate the Expectation using the Central Limit Theorem.

The Algorithm is the following

1. Let X be a random variable, simulate $(X_1, \ ..., X_N)$ a sample drawn along the law of X;
2. Compute estimators of the expectation and the variance

$$\mu_N = \frac{1}{N}\sum_{i=1}^{N} X_i \qquad\qquad \sigma_N^2 = \frac{1}{N-1}\sum_{i=1}^{N}(X_i - \mu_N)^2$$

3. Then, μ_N is an estimation of the expectation with an interval of confidence $I_{\alpha,N}$ and a level of confidence α given by:

$$I_{\alpha,N} = \left[\mu_N - c_\alpha \frac{\sigma_n}{\sqrt{N}}, \mu_N + c_\alpha \frac{\sigma_n}{\sqrt{N}}\right],$$

and for $\alpha = 95\%$, one has $c_\alpha \simeq 1.96$.

Application: computation of the expectation, using the Milstein Method

Tolerance for the computation of the expectation: 6.10^{-4}, level of confidence: 95%.

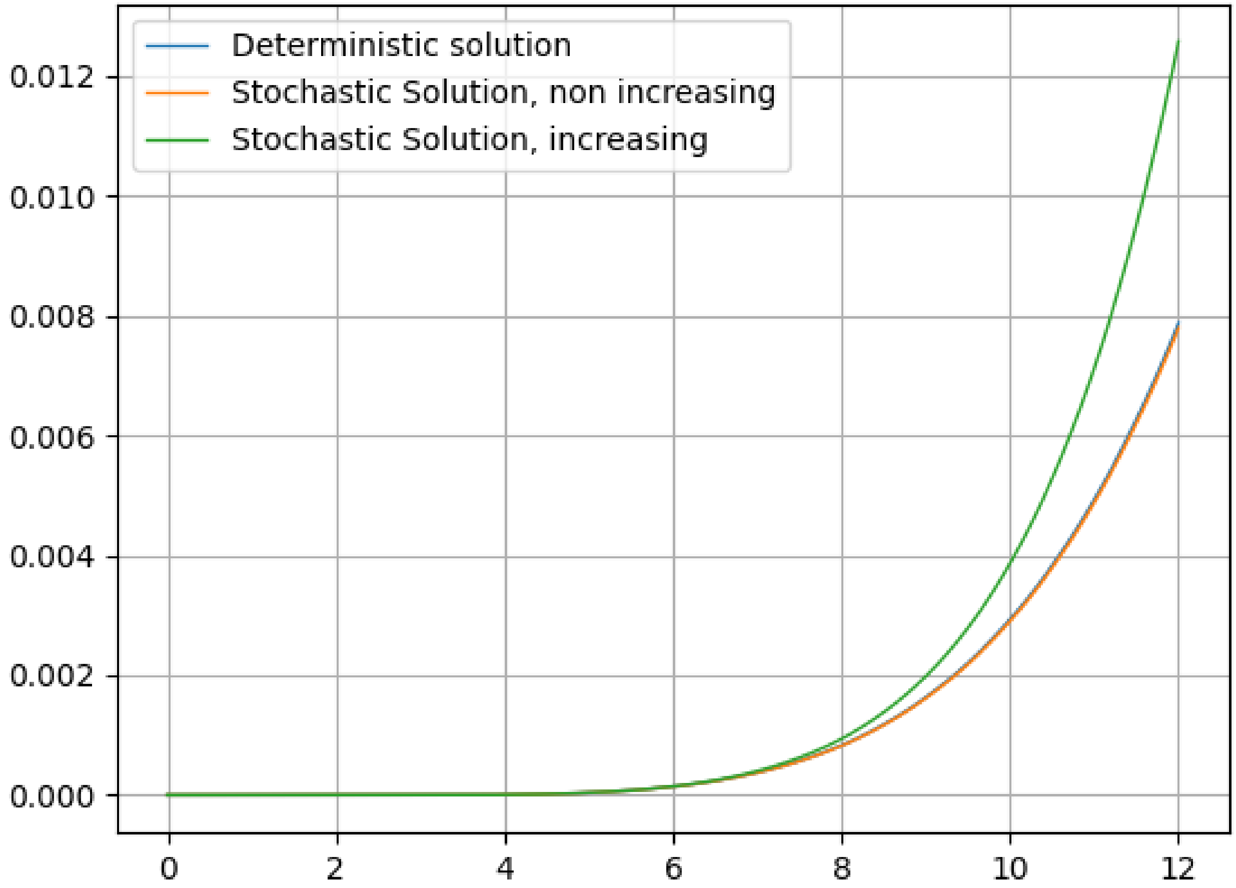


Figure 6: Comparison (expectation) between non increasing and increasing modelings. Data: $\beta = 0.5$, $h(\omega) \equiv 10^{-1}$, $\delta t = 10^{-2}$s., $T = 12$s.

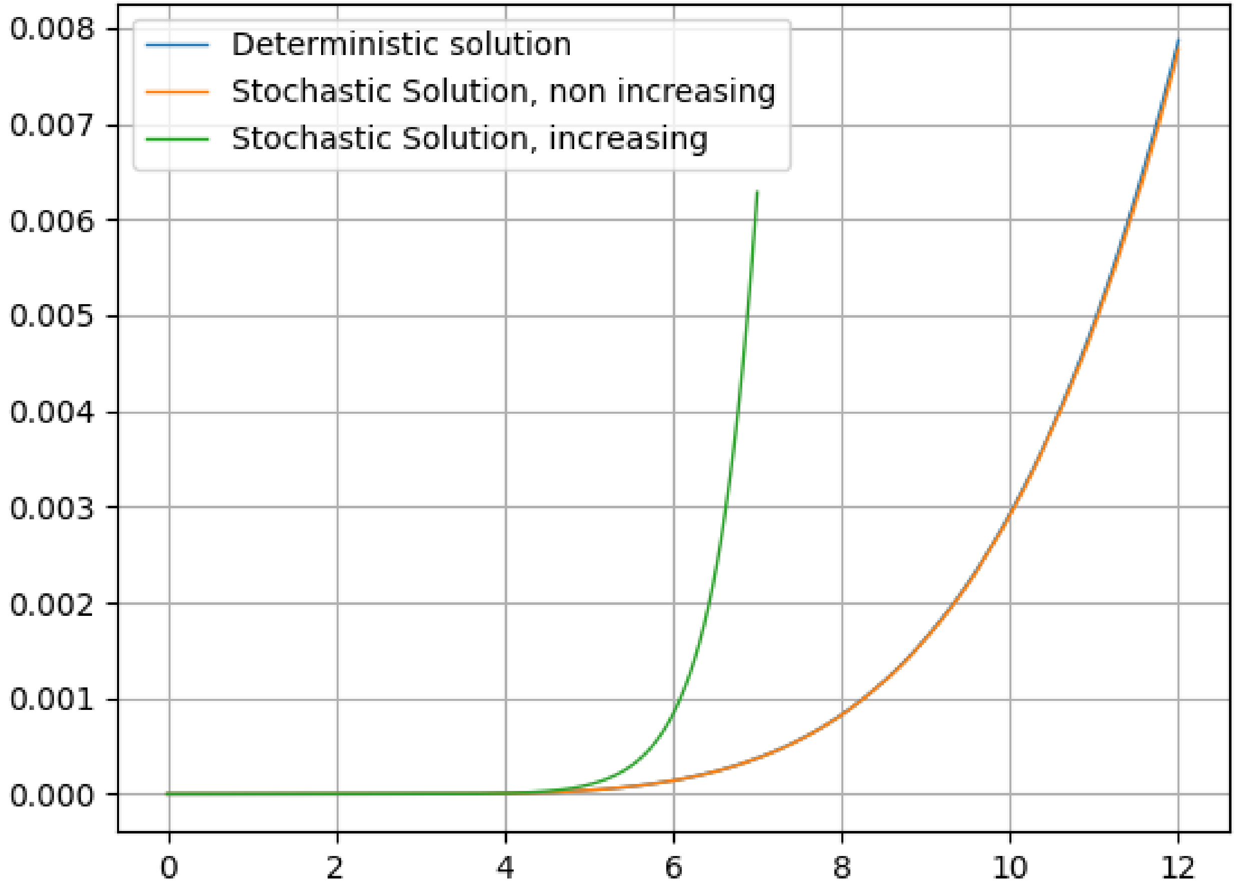


Figure 7: Expectation, $h(\omega) \equiv 5 \cdot 10^{-1}$, $\beta = 0.5$, $\delta t = 10^{-2}$s., $T = 12$s.

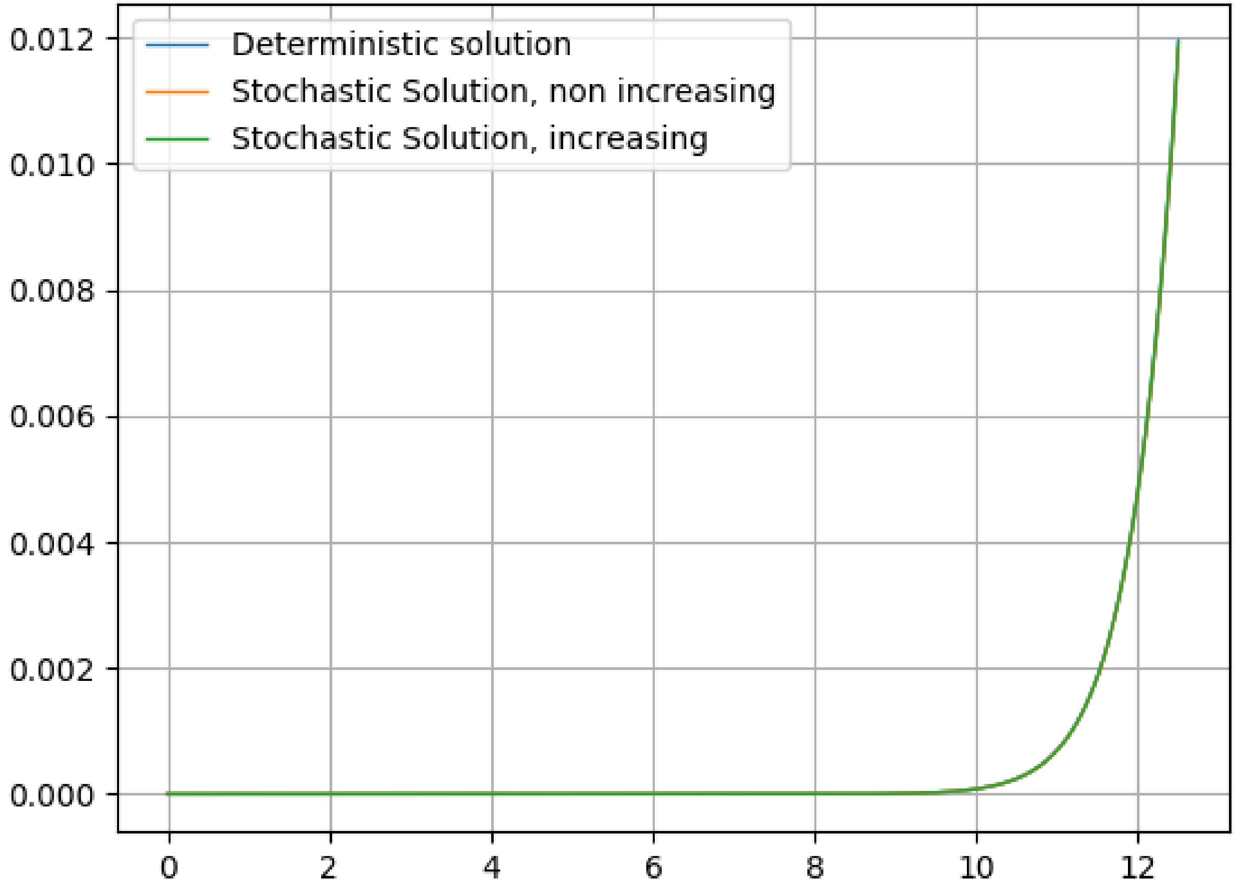


Figure 8: Expectation, $\beta = 0.1$, $h(\omega) \equiv 10^{-1}$, $\delta t = 10^{-2}$s., $T = 12.5$s.

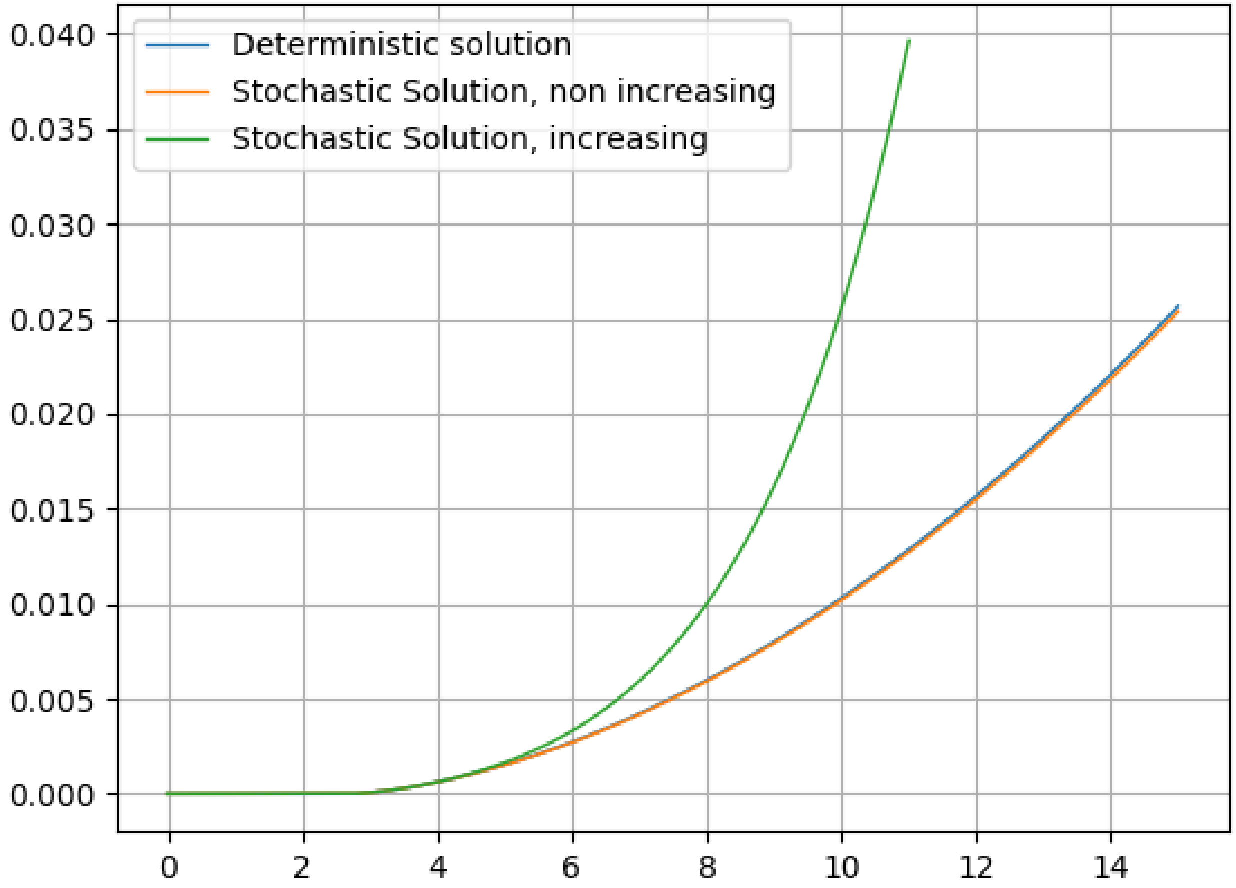


Figure 9: Expectation, $\beta = 2$, $h(\omega) \equiv 10^{-1}$, $\delta t = 10^{-2}$s., $T = 11$s.

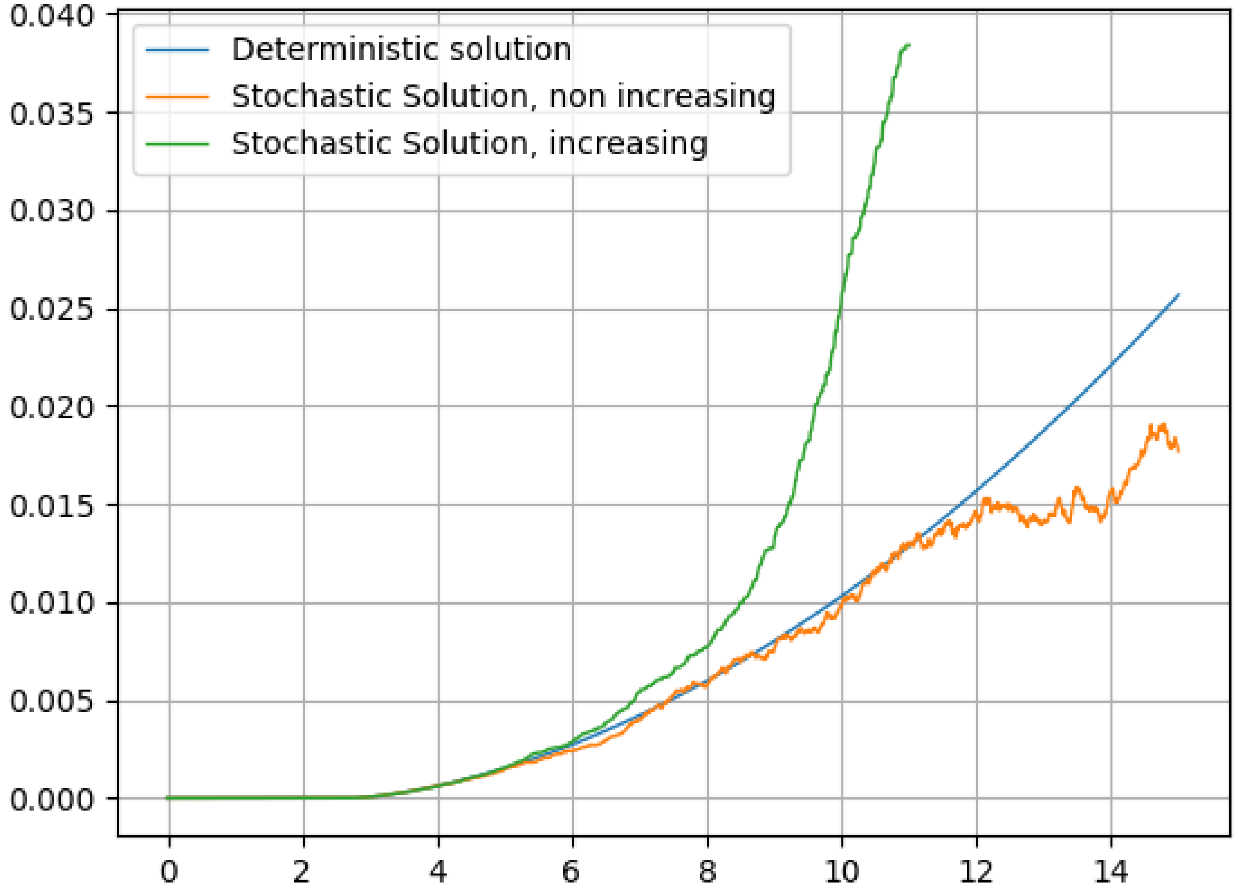
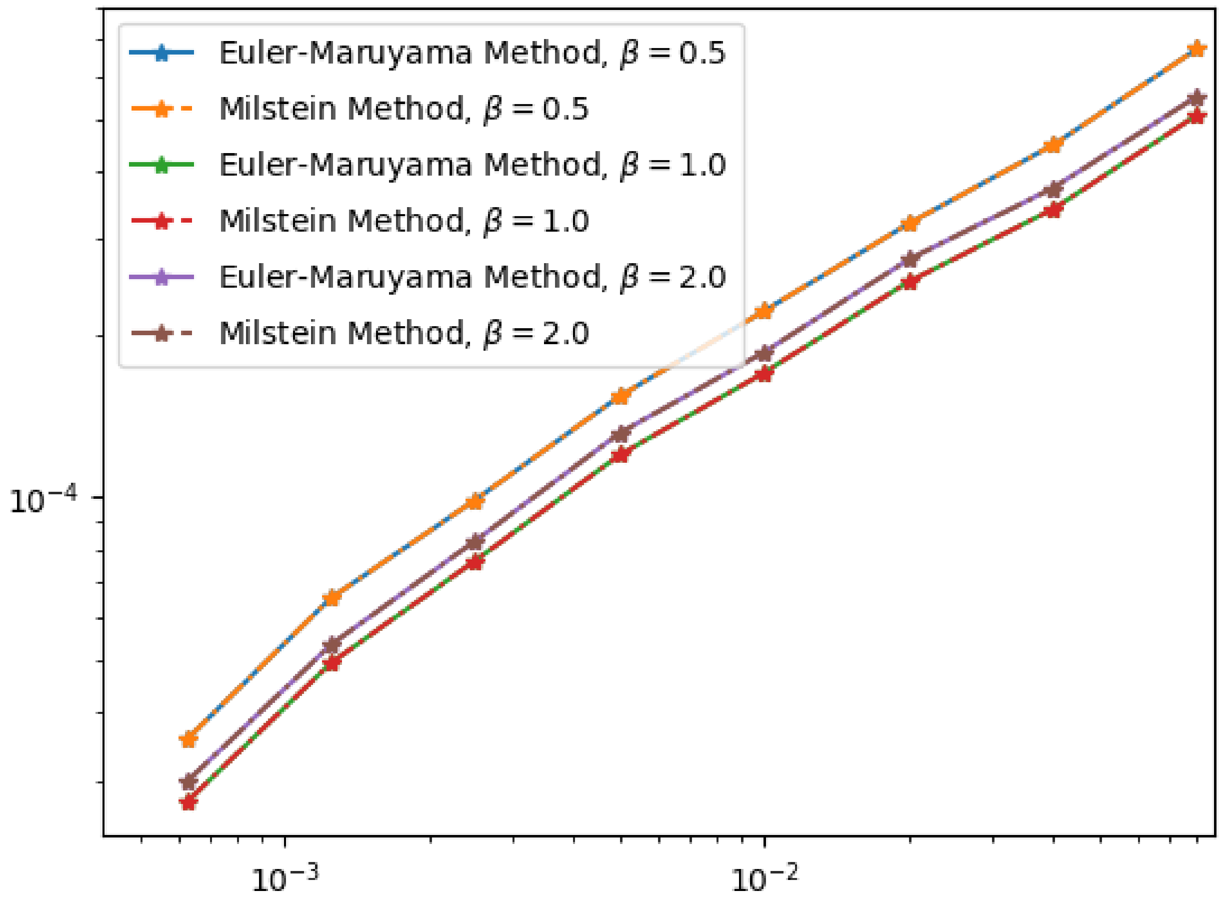


Figure 10: A solution, $\beta = 2$, $h(\omega) \equiv 10^{-1}$, $\delta t = 10^{-2}$s., $T = 11$s.

Convergence

We plot here the quantity $\mathbb{E}(\|u_{\delta t} - u_{ex}\|_{L^2(0,T)})$ w.r.t. δt.
u_{ex} is approximated by $u_{\delta t}$ with δt very small.

Result in the case of non strictly increasing stochastic solution



Result in the case of strictly increasing stochastic solution

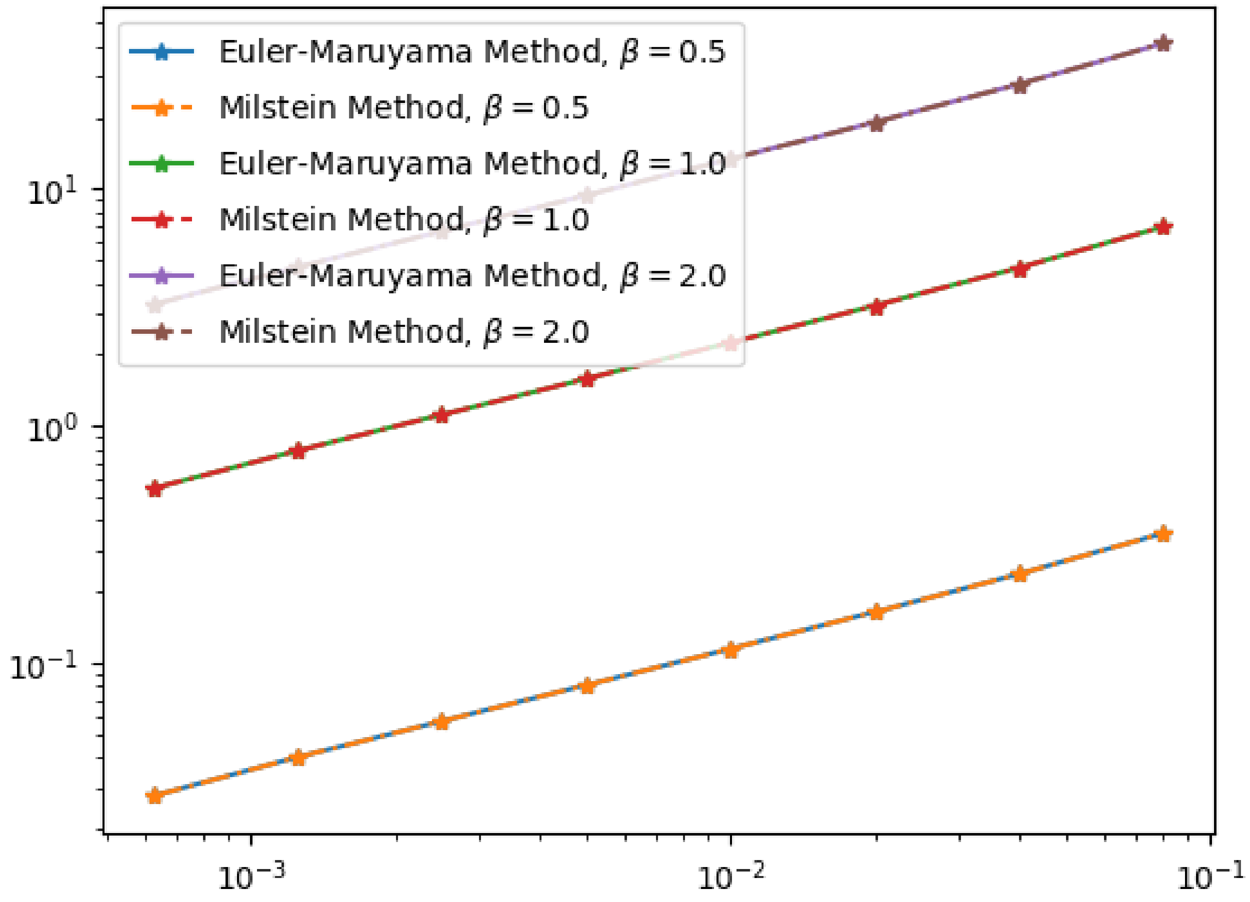
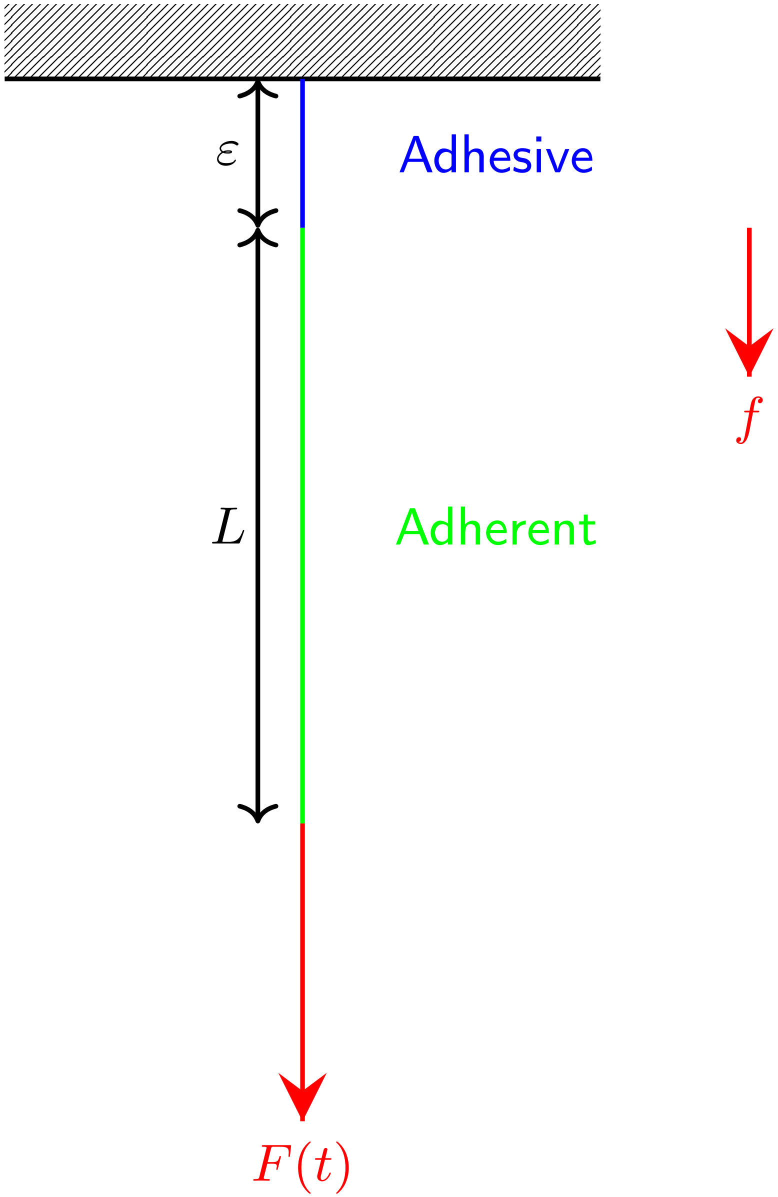


Figure 12: Strong convergence, $h(\omega) \equiv 10^{-1}$, $T = 10$s.

A second example (1D, quasi-static)



Remark

If the damage R is known, a analytic solution can be obtained, both in the case of 2 phases solution and with the interface law. The difference between the two solutions are controlled by ε^2.

Data

- $\bar{\omega} = 0.06$ Pa; $\eta(R) = \bar{\eta} = 3.6 \cdot 10^2$ Pa.
- $\varepsilon = 10^{-2}$cm., $L = 1$cm.;
- Young's modulus of the adherent: $E = 1$Pa;
- Young's modulus of the undamaged adhesive : $E_0 = 1$Pa;
- Normalized gravity: $f = 1$N, External force: $F(t) = 1 + 0.1\sin(t)$N.

Evolution of the damage

In the case where the interface law in considered, the damage verify

$$\dot{R}^{\beta} = -\frac{1}{3.6\cdot 10^2}\left(-0.06 - \frac{\pi}{E_0(1-2\pi R)^2}(\varepsilon + 0.1\sin(t))^2\right)_- \tag{6.1}$$

In the case of a 2 phases modeling, the damage verify

$$\dot{R}^{\beta} = -\frac{1}{3.6\cdot 10^2}\left(-0.06 - \frac{\pi}{E_0(1-2\pi R)^2}(\varepsilon + 0.1\sin(t))^2 - \frac{1}{2}E_0(1-2\pi R)\varepsilon^2\right)_- \tag{6.2}$$

Comparison of the damage obtained with the 2 modelings

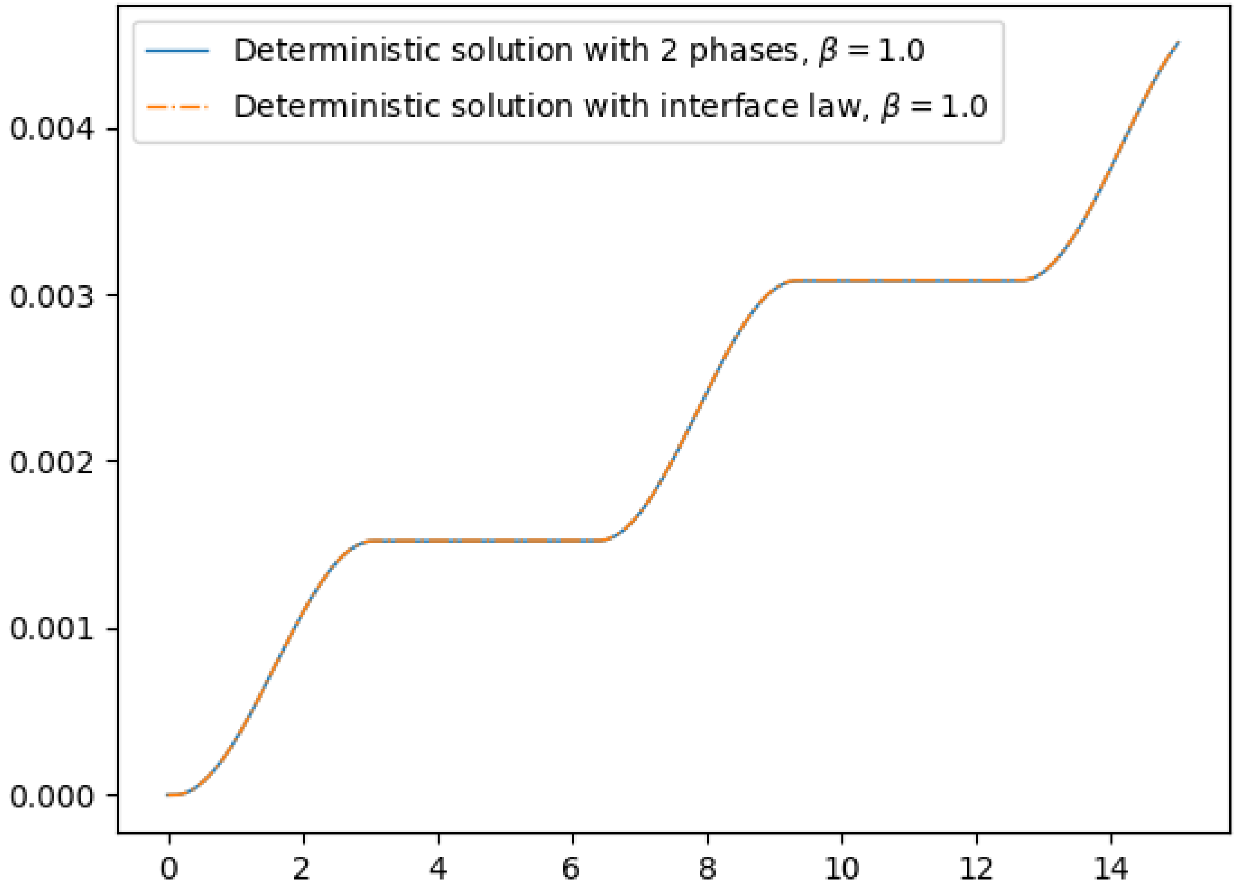


Figure 13: Comparison of the damage obtained with the 2 modelings ($\beta = 1$).

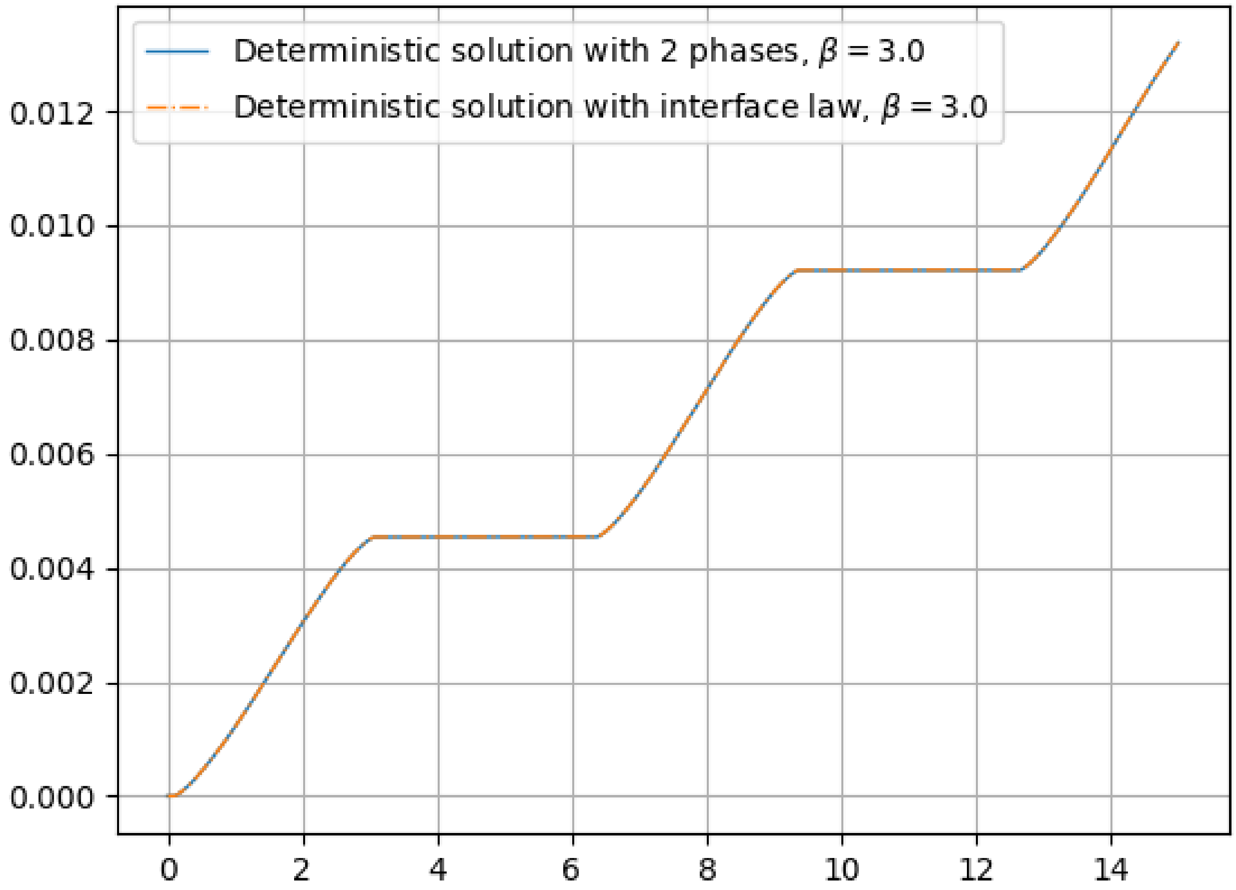


Figure 14: Comparison of the damage obtained with the 2 modelings ($\beta = 3$).

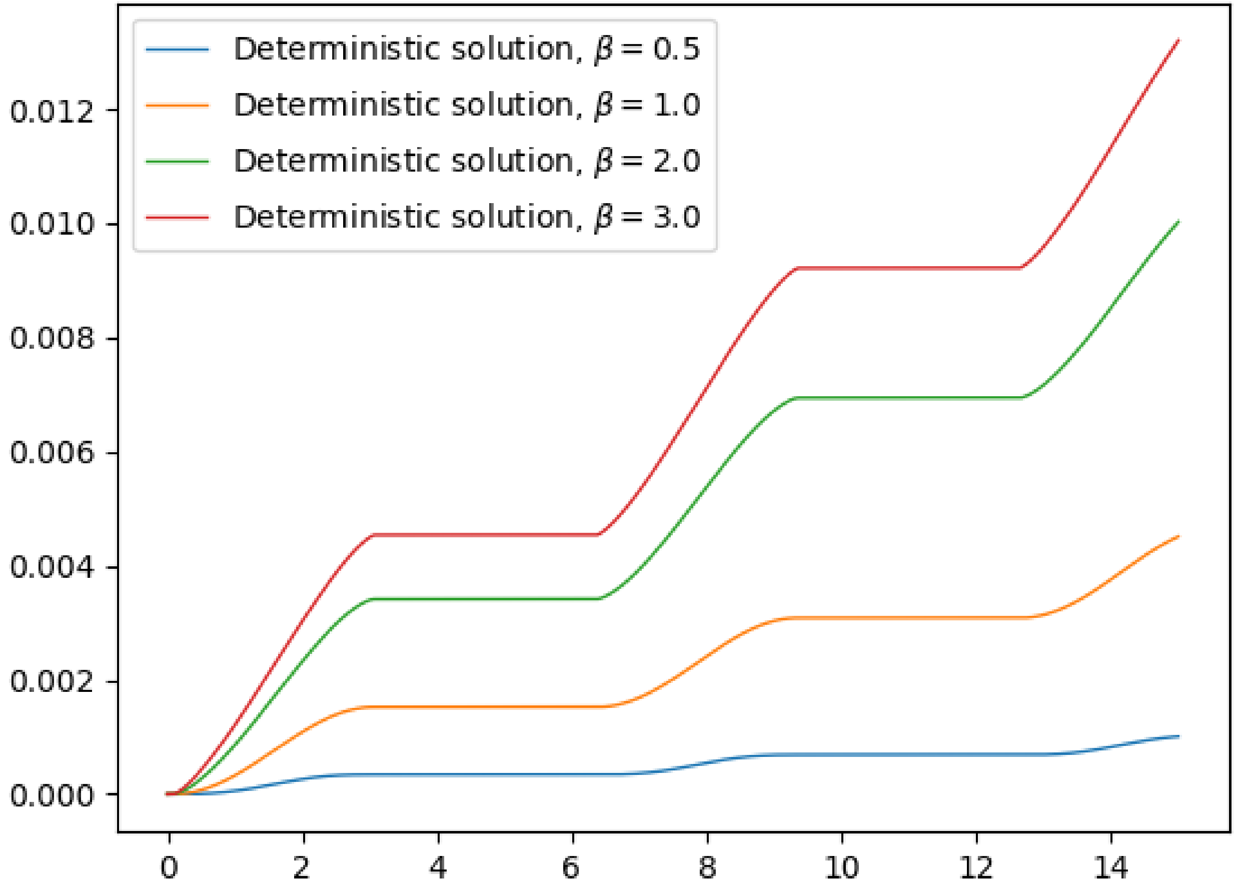


Figure 15: Evolution of the damage with various values of β (deterministic case with an interface law).

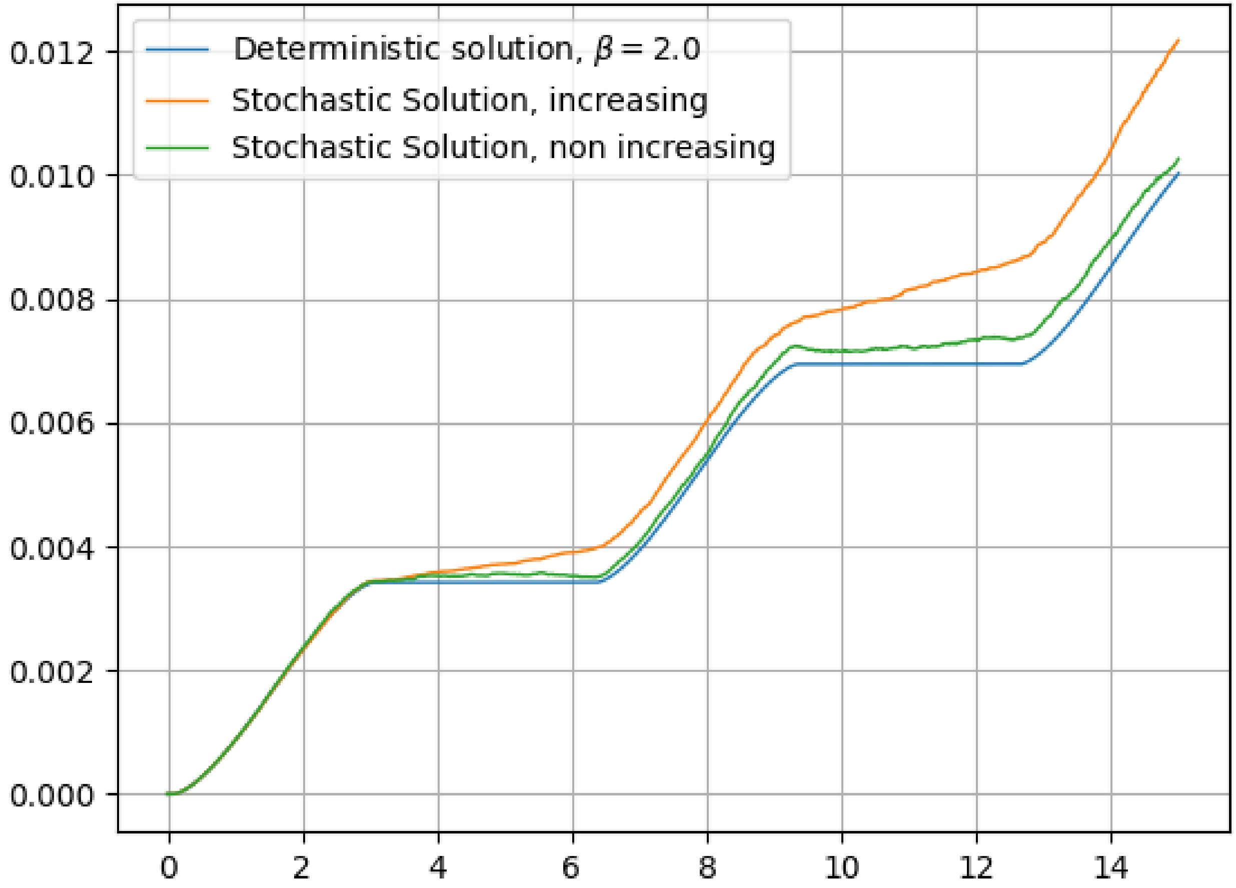


Figure 16: Comparison of the evolution of the damage with the deterministic and the stochastic modelings (1 solution, $\beta = 2$, with an interface law).

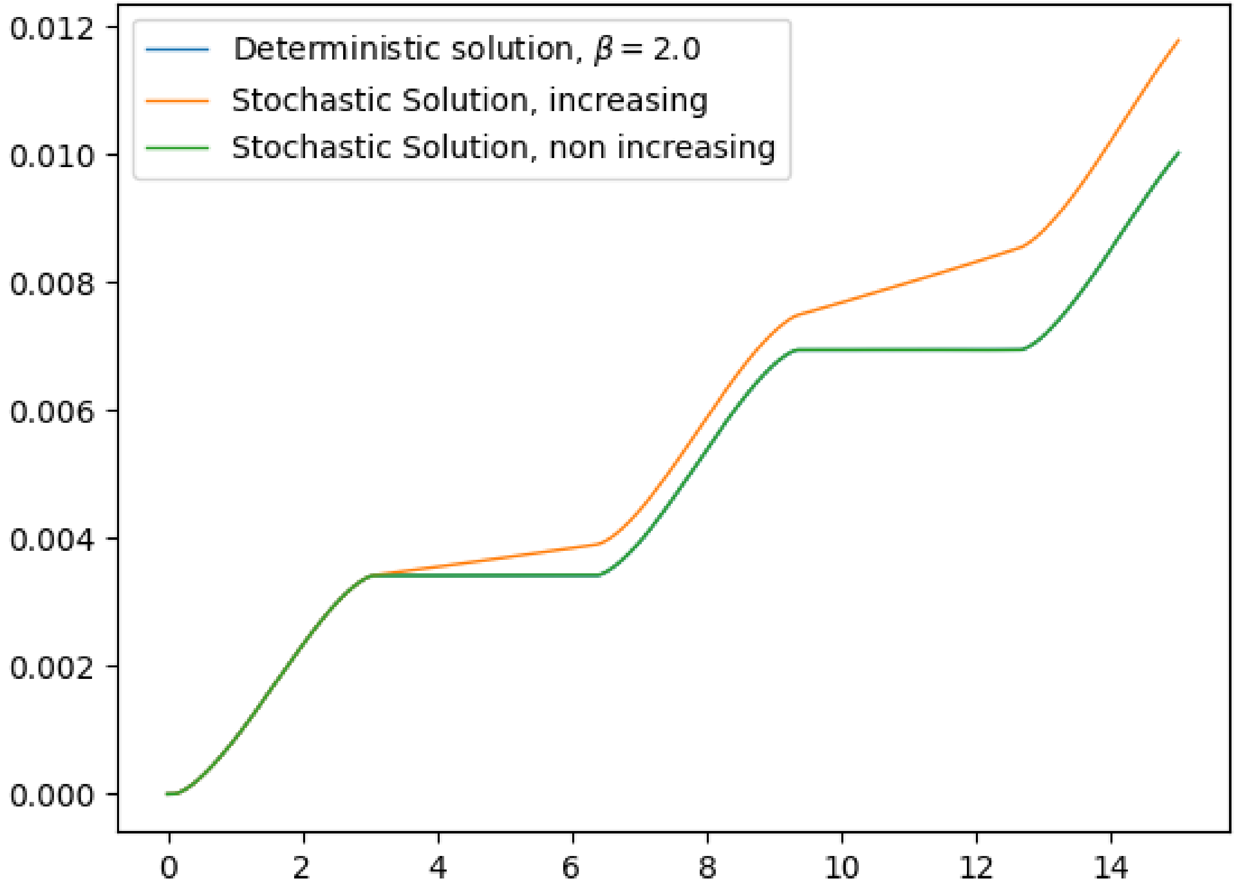


Figure 17: Comparison of the evolution of the damage with the deterministic and the stochastic modelings (Average, $\beta = 2$, with an interface law).

Thank you for your attention !

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