Modelling and analysis of surface damage problems

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Contact problems with adhesion

Applications to

- machine designing and manufacturing (use of **adhesive materials** in automotive and aerospace industry...)
- use of **layered composite structures** in building and civil engineering

• the interface regions between laminates affect the strength and stability of the structural elements
• the degradation of the adhesive substance on such regions may lead to material failure

$$\Downarrow$$

surface damage models

[Frémond, '80s–'90s & "Non-smooth thermomechanics" 2002]

$\rightsquigarrow$ **energy and dissipation concentrated** on the contact surface

The model

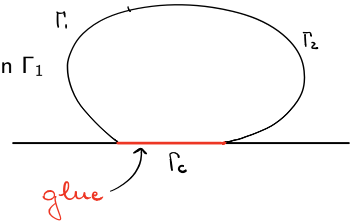
We consider
a thermoviscoelastic body $\Omega \subset \mathbb{R}^3$ which is in

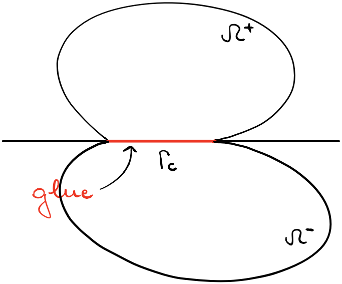
contact with adhesion

with a rigid support on a **(flat)** prescribed part Γ_c of its boundary

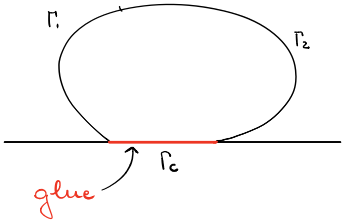
$$\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_c$$

- assigned displacement on Γ_1
- assigned traction on Γ_2





Ω^+
Γ_c
glue
Ω^-



we study its evolution taking into account

- **viscoelastic** behavior in the bulk domain (under small deformations assumption)
- **the unilateral contact** (Signorini conditions)
- **the adhesion** ($\sim$ glue) between the body and the support
- **frictional effects** (Coulomb law)
- **thermal effects**: in the bulk domain and on the contact surface (for the moment, **neglected**)

Related literature

(on static, quasistatic, dynamic **contact problems** with or without friction, with or without adhesion/delamination, mainly in the **isothermal case**):

- ....
- Ballard, Cocou, Jean, Lebon, Léger, Point, Pratt, Raous
- Andersson, Andrews, Klarbring, Kuttler, Shillor, Wright, Sofonea, Telega
- Martins, Monteiro Marques, Oden
- Migórski, Mantic, Kruzič, Panagiotopoulos
- Bock, Eck, Jarušek, Krbec, Schatzman
- Kočvara, Mielke, Roubiček, Thomas
- ....

The model: the variables

♣ In the **isothermal case**

- in the **bulk domain** Ω:

$\varepsilon(\mathbf{u})$ symm. linearized strain tensor

($\mathbf{u}$ small displacement)

- on the **contact surface** Γ_c:

χ (scalar) **adhesion** parameter

"phase parameter" $\sim$ proportion of active bonds between body & support

The equations for u and χ

♣ Equations for **u** and χ are recovered from the **principle of virtual powers**

♣ The **energy balance** of the system also includes **micro-forces and micro-motions**, according to M. Frémond's approach

- **momentum balance:**

$$\begin{cases} -\operatorname{div}\sigma = \mathbf{f} \quad \text{in } \Omega \times (0,T), \\ \begin{cases} \sigma\mathbf{n} = \mathbf{R} \quad \text{in } \Gamma_c \times (0,T), \\ \mathbf{u} = \mathbf{0} \quad \text{in } \Gamma_1 \times (0,T), \\ \sigma\mathbf{n} = \mathbf{g} \quad \text{in } \Gamma_2 \times (0,T), \end{cases} \end{cases}$$

$$\begin{cases} \sigma \text{ stress tensor} \\ \mathbf{R} \text{ reaction on the contact surface} \\ \mathbf{f} \text{ volume force, } \mathbf{g} \text{ traction} \end{cases}$$

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- **balance equation for microscopic motions:**

$$\begin{cases} B - \operatorname{div}_s \mathbf{H} = a \ \text{ in } \Gamma_c \times (0,T), \\ \mathbf{H} \cdot \mathbf{n}_s = 0 \ \text{ on } \partial\Gamma_c \times (0,T), \end{cases}$$

$$\begin{cases} B, \ \mathbf{H} \text{ microscopic internal forces} \\ a \text{ microscopic external source} \end{cases}$$

Energy and dissipation functionals

Constitutive laws for σ, $\mathbf{R}, B, \mathbf{H}$ are given in terms of **volume & surface free energies**

$$\Psi_\Omega = \Psi_\Omega(\varepsilon(\mathbf{u})), \qquad \Psi_{\Gamma_c} = \Psi_{\Gamma_c}(\mathbf{u}_{|\Gamma_c}, \chi, \nabla\chi)$$

and the **volume & surface potentials of dissipation**

$$\Phi_\Omega = \Phi_\Omega(\varepsilon(\dot{\mathbf{u}})), \qquad \Phi_{\Gamma_c} = \Phi_{\Gamma_c}(\dot{\chi}, \dot{\mathbf{u}}_{|\Gamma_c})$$

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- **[Raous, Cangémi, Cocou, '99]** for a model close to the present one;
- **[Del Piero, Raous, '10]** for general models coupling friction, adhesion and unilateral contact.

The adhesion phenomenon

Notation for the normal and tangential components of displacement vector $\mathbf{u}$ and stress vector $\sigma\mathbf{n}$

$$\mathbf{u} = u_N\mathbf{n} + \mathbf{u}_T, \quad u_N = u_i n_i, \quad \sigma\mathbf{n} = \sigma_N\mathbf{n} + \sigma_T, \quad \sigma_N = \sigma_{ij} n_i n_j$$

with $\mathbf{n} = (n_i)$ outward normal unit vector to $\partial\Omega$.

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with $\mathbf{n} = (n_i)$ outward normal unit vector to $\partial\Omega$.

• The "surface damage parameter" χ ~ **fraction of active glue fibers** at each point of the contact surface

- $\chi = 0$ no adhesion (completely broken bonds)
- $\chi = 1$ complete adhesion (unbroken bonds)
- $0 < \chi < 1$ partial adhesion

We have to enforce

$$\boxed{\chi \in [0, 1]}$$

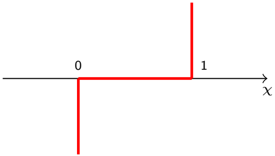
The adhesion phenomenon

We impose the constraint $\boxed{\chi \in [0,1]}$ by the term $I_{[0,1]}(\chi)$ in the surface energy functional

$$\Psi_{\Gamma_c} = \frac{1}{2}\chi|\mathbf{u}|^2 + I_{(-\infty,0]}(\mathbf{u}_N) + \omega(1-\chi) + \frac{1}{2}|\nabla_s \chi|^2 + I_{[0,1]}(\chi)$$

$\Rightarrow$ $\boxed{\partial I_{[0,1]}(\chi) \text{ in eq. for } \chi}$

$$\partial I_{[0,1]}(\chi) = \begin{cases} (-\infty, 0], & \text{if } \chi = 0, \\ 0, & \text{if } 0 < \chi < 1, \\ [0, +\infty), & \text{if } \chi = 1. \end{cases}$$



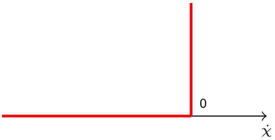
The adhesion phenomenon

- If the "damage" of the glue is **irreversible**, we enforce $\boxed{\dot\chi \leq 0}$ by the term $I_{(-\infty,0]}(\dot\chi)$ in the surface dissipation potential

$$\Phi_{\Gamma_c} = \frac{1}{2}|\dot\chi|^2 + I_{(-\infty,0]}(\dot\chi)$$

$\Rightarrow$ $\boxed{\partial I_{(-\infty,0]}(\dot\chi) \text{ in eq. for } \chi}$

$$\partial I_{(-\infty,0]}(\dot\chi) = \begin{cases} 0, & \text{if } \dot\chi < 0, \\ [0,+\infty), & \text{if } \dot\chi = 0. \end{cases}$$



The PDE system: the evolution of the adhesion

The evolution of the adhesion on the contact surface is ruled by

$$\dot{\chi} - \Delta_s \chi + \partial I_{[0,1]}(\chi) + \partial I_{]-\infty,0]}(\dot{\chi}) \ni \omega - \frac{1}{2}|\mathbf{u}|^2 \qquad \text{on } \Gamma_c \times (0,T)$$

$$\partial_{\mathbf{n_s}} \chi = 0, \qquad \text{on } \partial\Gamma_c \times (0,T)$$

- $\partial I_{[0,1]}(\chi) \Rightarrow \chi \in [0,1]$ **(physical consistency)**
- $\partial I_{]-\infty,0]}(\dot{\chi}) \Rightarrow \dot{\chi} \leq 0$ **(irreversible adhesion)**
- $\omega > 0$ constant (coefficient of internal cohesion)
- $-\frac{1}{2}|\mathbf{u}|^2$ source of damage due to displacement.

The unilateral contact

- The **normal reaction** on Γ_c has to ensure the **impenetrability condition**

$$u_N \leq 0 \quad \text{on } \Gamma_c$$

and to render the **Signorini conditions**.

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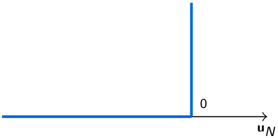
It is given by

$$R_N \in -\frac{\partial \Psi_{\Gamma_c}}{\partial \mathbf{u}_N}$$

that is

$$\boxed{R_N = \sigma_N \in -\chi u_N - \partial I_{]-\infty,0]}(u_N)}$$

$$\partial I_{(-\infty,0]}(\mathbf{u}_N) = \begin{cases} 0, & \text{if } \mathbf{u}_N < 0, \\ [0,+\infty), & \text{if } \mathbf{u}_N = 0. \end{cases}$$



Signorini conditions

$$\boxed{\sigma_N \in -\chi u_N - \partial I_{]-\infty,0]}(u_N)} \Leftrightarrow \boxed{u_N \leq 0, \;\; \sigma_N + \chi u_N \leq 0, \;\; u_N\,(\sigma_N + \chi u_N) = 0}$$

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- If $\chi = 0$ (no adhesion) then $\boxed{u_N \leq 0,\ \ \sigma_N \leq 0,\ \ u_N\,\sigma_N = 0}$

(classical **Signorini conditions**)

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- If the adhesion is **active** $\chi > 0$

$$\sigma_N \in -\chi u_N - \partial I_{]-\infty,0]}(u_N)$$

i.e., there is a reaction **counteracting separation**:

$$\sigma_N = -\chi u_N > 0 \text{ if } u_N < 0$$

The friction effects: the Coulomb law

The **tangential component** of the reaction on Γ_c is given by

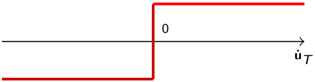
$$\mathbf{R}_T \in -\frac{\partial \Phi_{\Gamma_c}}{\partial \dot{\mathbf{u}}_T}$$

that is
$$\mathbf{R}_T = \sigma_T \in -\chi \mathbf{u}_T - \nu |\sigma_N + \chi u_N| \mathbf{d}(\dot{\mathbf{u}}_T)$$

where

$$\mathbf{d}(\mathbf{v}_T) = \begin{cases} \dfrac{\mathbf{v}_T}{|\mathbf{v}_T|} & \text{if } \mathbf{v}_T \neq \mathbf{0} \\ \mathbf{z}_T \quad |\mathbf{z}| \leq 1 & \text{if } \mathbf{v}_T = \mathbf{0} \end{cases}$$

$\rightsquigarrow$ if v_T is scalar, then $\mathbf{d} = \mathrm{Sign} : \mathbb{R} \rightrightarrows \mathbb{R}$



- ν **friction coefficient**
- $\sigma_N + \chi u_N \in -\partial I_{(-\infty,0]}(u_N)$

The friction effects: the Coulomb law

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The regularized (nonlocal) Coulomb law

The tangential component of the reaction needs to be **regularized**

$$\sigma_T \in -\chi \mathbf{u}_T - \nu |\mathcal{R}(\sigma_N + \chi u_N)| \mathbf{d}(\dot{\mathbf{u}}_T)$$

where

- $$\mathbf{d}(\mathbf{v}_T) = \begin{cases} \dfrac{\mathbf{v}_T}{|\mathbf{v}_T|} & \text{if } \mathbf{v}_T \neq \mathbf{0} \\ \mathbf{z}_T \quad |\mathbf{z}| \leq 1 & \text{if } \mathbf{v}_T = \mathbf{0} \end{cases}$$
- $\mathcal{R}$ **nonlocal smoothing operator** (physically meaningful)

For friction problems without adhesion, use of $\mathcal{R}$ first proposed in **[Duvaut,'80]**

Nonlocal Coulomb law for unilateral contact

$$\sigma_T \in -\chi \mathbf{u}_T - \nu|\mathcal{R}(\sigma_N + \chi u_N)|\mathbf{d}(\dot{\mathbf{u}}_T)$$

generalizes the **nonlocal** Coulomb law, accounting for adhesion

$$\begin{aligned}
&|\sigma_T + \chi \mathbf{u}_T| \leq \nu|\mathcal{R}(\sigma_N + \chi u_N)|,\\
&|\sigma_T + \chi \mathbf{u}_T| < \nu|\mathcal{R}(\sigma_N + \chi u_N)| \Longrightarrow \dot{\mathbf{u}}_T = \mathbf{0},\\
&|\sigma_T + \chi \mathbf{u}_T| = \nu|\mathcal{R}(\sigma_N + \chi u_N)| \Longrightarrow \exists \lambda \geq 0 : \quad \dot{\mathbf{u}}_T = -\lambda(\sigma_T + \chi \mathbf{u}_T)
\end{aligned}$$

The Problem: variational formulation

- Bilinear forms of linear viscoelasticity

$$\begin{cases} a(\mathbf{u},\mathbf{v}) := \int_\Omega a_{ijkh}\varepsilon_{kh}(\mathbf{u})\varepsilon_{ij}(\mathbf{v}), \\ b(\mathbf{u},\mathbf{v}) = \int_\Omega b_{ijkh}\varepsilon_{kh}(\mathbf{u})\varepsilon_{ij}(\mathbf{v}) \end{cases}$$

for $\mathbf{u}$, $\mathbf{v} \in \mathbf{W} = \{\mathbf{v} \in (H^1(\Omega))^3 : \mathbf{v} = \mathbf{0}$ a.e. on $\Gamma_1\}$.

- **The problem:** Find $(\mathbf{u}, \chi, \eta)$ such that

$$b(\dot{\mathbf{u}},\mathbf{v}) + a(\mathbf{u},\mathbf{v}) + \int_{\Gamma_c} \chi\mathbf{u}\cdot\mathbf{v} + \int_{\Gamma_c} \eta\mathbf{v}\cdot\mathbf{n} + \int_{\Gamma_c} \nu|\mathcal{R}(-\eta)|\mathbf{d}(\dot{\mathbf{u}}_T)\cdot\mathbf{v}_T \ni \langle\mathbf{F},\mathbf{v}\rangle$$

$$\forall\mathbf{v}\in\mathbf{W} \quad \text{a.e. in } (0,T)$$

$$\eta \in \partial I_{(-\infty,0]}(u_N) \quad \text{on } \Gamma_c \times (0,T)$$

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$$\dot{\chi} - \Delta_s\chi + \partial I_{(-\infty,0]}(\dot{\chi}) + \partial I_{[0,1]}(\chi) \ni \omega - \frac{1}{2}|\mathbf{u}|^2 \quad \text{on } \Gamma_c \times (0,T),$$

$$\partial_{\mathbf{n}_s}\chi = 0 \quad \text{on } \partial\Gamma_c \times (0,T) \qquad + \textit{Cauchy conditions}$$

Analytical difficulties

$$b(\dot{\mathbf{u}}, \mathbf{v}) + a(\mathbf{u}, \mathbf{v}) + \int_{\Gamma_c} \chi \mathbf{u}\mathbf{v} + \int_{\Gamma_c} \eta \mathbf{v} \cdot \mathbf{n} +$$

$$+ \int_{\Gamma_c} \nu |\mathcal{R}(-\eta)| \mathbf{d}(\dot{\mathbf{u}}_T) \cdot \mathbf{v}_T \ni \langle \mathbf{F}, \mathbf{v} \rangle \quad \forall \mathbf{v} \in \mathbf{W} \ \text{ a.e. in } (0, T)$$

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$\rightsquigarrow$ **double multivalued constraint** on χ and $\dot{\chi}$

$\Rightarrow$ **doubly nonlinear** character

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⇝ **(quadratic) coupling** terms on the **boundary**

⇒ (we need **sufficient regularity** for $\mathbf{u}$ and $\dot{\mathbf{u}}$ to control their traces)

Analytical difficulties

$$b(\dot{\mathbf{u}},\mathbf{v})+a(\mathbf{u},\mathbf{v})+\int_{\Gamma_c}\chi\mathbf{u}\mathbf{v}+\int_{\Gamma_c}\eta\mathbf{v}\cdot\mathbf{n}$$

$$+\int_{\Gamma_c}\nu|\mathcal{R}(-\eta)|\mathbf{d}(\dot{\mathbf{u}}_T)\cdot\mathbf{v}_T\ni\langle\mathbf{F},\mathbf{v}\rangle\quad\forall\mathbf{v}\in\mathbf{W}\ \text{ a.e. in } (0,T)$$

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$$\partial_{\mathbf{n}_s}\chi=0\quad\text{ on } \partial\Gamma_c\times(0,T)\qquad + \textit{Cauchy conditions}$$

$\rightsquigarrow$ **double multivalued constraint** on u_N and $\dot{\mathbf{u}}_T$ on the boundary.

$\Rightarrow$ **main difficulty!**

A regularization of the boundary term $|\mathcal{R}(-\eta)|\mathbf{d}(\dot{\mathbf{u}}_T)$ is **crucial!**

A global-in-time existence result

$$b(\dot{\mathbf{u}}, \mathbf{v}) + a(\mathbf{u}, \mathbf{v}) + \int_{\Gamma_c} \chi \mathbf{u}\mathbf{v} + \int_{\Gamma_c} \eta \mathbf{v} \cdot \mathbf{n} +$$

$$+ \int_{\Gamma_c} \nu |\mathcal{R}(-\eta)| \mathbf{d}(\dot{\mathbf{u}}_T) \cdot \mathbf{v}_T \ni \langle \mathbf{F}, \mathbf{v} \rangle \quad \forall \mathbf{v} \in \mathbf{W} \text{ a.e. in } (0, T)$$

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$$\partial_{\mathbf{n}_s} \chi = 0 \quad \text{on } \partial\Gamma_c \times (0, T) \quad + \text{Cauchy conditions}$$

There **exists a solution** $(\mathbf{u}, \chi, \eta)$

$$\mathbf{u} \in H^1(0, T; H^1(\Omega))$$

$$\chi \in W^{1,\infty}(0, T; L^2(\Gamma_c)) \cap H^1(0, T; H^1(\Gamma_c)) \cap L^\infty(0, T; H^2(\Gamma_c))$$

$$\eta \in L^2(0, T; H^{-1/2}(\Gamma_c))$$

Outline of the proof of existence

- Moreau-Yosida regularization of non-smooth operators
- Time discretization scheme (time-incremental minimization)
- Existence result for the discretized system
- Uniform estimates
- Passage to the limit
  - Identification of nonlinearities

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- **Uniform estimates**
- **Passage to the limit**
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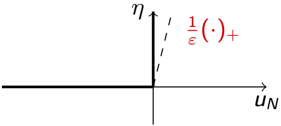
The approximate problem

Approximation: Moreau-Yosida regularization of the multivalued operators

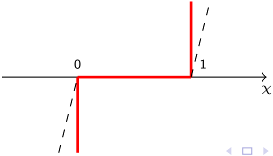
$$\begin{cases} \partial I_{(-\infty,0]}(u_N) & \text{replaced by} \quad (\partial I_{(-\infty,0]})_\varepsilon(u_N) \\ \partial I_{[0,1]}(\chi) & \text{replaced by} \quad (\partial I_{[0,1]})_\varepsilon(\chi) \end{cases}$$

$\twoheadleftarrow$ **normal compliance**

- $\eta \in \partial I_{(-\infty,0]}(u_N) \rightsquigarrow \eta_\varepsilon = (\partial I_{(-\infty,0]})_\varepsilon(u_N) = \frac{1}{\varepsilon}(u_N)_+$



- $\partial I_{[0,1]}(\chi) \rightsquigarrow (\partial I_{[0,1]})_\varepsilon(\chi)$



The approximate problem

$$b(\dot{\mathbf{u}},\mathbf{v}) + a(\mathbf{u},\mathbf{v}) + \int_{\Gamma_c} \chi \mathbf{u}\cdot\mathbf{v} + \int_{\Gamma_c} \eta \mathbf{v}\cdot\mathbf{n} + + \int_{\Gamma_c} |\mathcal{R}(-\eta)| \mathbf{d}(\dot{\mathbf{u}}_T) \cdot \mathbf{v}_T \ni \langle \mathbf{F},\mathbf{v}\rangle$$

for all $\mathbf{v} \in W$ a.e. in $(0,T)$

$$\eta \in \partial I_{(-\infty,0]}(u_N) \quad \text{on } \Gamma_c \times (0,T)$$

$$\dot{\chi} - \Delta_s \chi + \partial I_{(-\infty,0]}(\dot{\chi}) + \partial I_{[0,1]}(\chi) \ni \omega - \frac{1}{2}|\mathbf{u}|^2 \quad \text{on } \Gamma_c \times (0,T),$$

$$\partial_{\mathbf{n}_s}\chi = 0 \quad \text{on } \partial\Gamma_c \times (0,T),$$

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$$b(\dot{\mathbf{u}},\mathbf{v}) + a(\mathbf{u},\mathbf{v}) + \int_{\Gamma_c} \chi \mathbf{u}\cdot\mathbf{v} + \int_{\Gamma_c} \eta_{\varepsilon}\mathbf{v}\cdot\mathbf{n} + + \int_{\Gamma_c} |\mathcal{R}(-\eta_{\varepsilon})|\mathbf{d}(\dot{\mathbf{u}}_T)\cdot\mathbf{v}_T \ni \langle \mathbf{F},\mathbf{v}\rangle$$

for all $\mathbf{v} \in W$ a.e. in $(0,T)$

$$\eta_{\varepsilon} = (\partial I_{(-\infty,0]})_{\varepsilon}(u_N) \quad \text{on } \Gamma_c \times (0,T)$$

$$\dot{\chi} - \Delta_s \chi + \partial I_{(-\infty,0]}(\dot{\chi}) + (\partial I_{[0,1]}(\chi))_{\varepsilon} \ni \omega - \frac{1}{2}|\mathbf{u}|^2 \quad \text{on } \Gamma_c \times (0,T),$$

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$$\partial_{\mathbf{n}_s}\chi = 0 \quad \text{on } \partial\Gamma_c \times (0,T),$$

- Time discretization scheme (time-incremental minimization)
- Existence result for the discretized system
- Uniform estimates

First a priori estimate

- **Energy estimate:**

$$\int_0^t \quad b(\dot{\mathbf{u}}, \dot{\mathbf{u}}) + a(\mathbf{u}, \dot{\mathbf{u}}) + \int_{\Gamma_c} (\chi \mathbf{u} \cdot \dot{\mathbf{u}} + \eta \dot{\mathbf{u}}_N + |\mathcal{R}(-\eta)| \mathbf{d}(\dot{\mathbf{u}}_T) \cdot \dot{\mathbf{u}}_T) \ni \langle \mathbf{F}, \dot{\mathbf{u}} \rangle$$

$$+$$

$$\int_0^t \int_{\Gamma_c} \quad \dot{\chi} - \Delta_s \chi + \partial I_{(-\infty,0]}(\dot{\chi}) + \partial I_{[0,1]}(\chi) \ni \omega - \frac{1}{2}|\mathbf{u}|^2 \quad \times \quad \dot{\chi}$$

- Some terms cancel out and we get

$$|\mathbf{u}|_{H^1(0,T;W)} \leq C$$
$$|\chi|_{H^1(0,T;L^2(\Gamma_c)) \cap L^\infty(0,T;H^1(\Gamma_c))} \leq C$$

In particular

$$|\mathbf{u}_{|\Gamma_c}|_{H^1(0,T;L^4(\Gamma_c))} \leq C$$

Second a priori estimate

- Formally

$$\int_0^t \int_{\Gamma_c} \quad \dot{\chi} - \Delta_s \chi + \partial I_{(-\infty,0]}(\dot{\chi}) + \partial I_{[0,1]}(\chi) \ni \omega - \frac{1}{2}|\mathbf{u}|^2 \quad \times \quad \partial_t(-\Delta\chi + \partial I_{[0,1]}(\chi))$$

- **Monotonicity** arguments + **integration by parts** in time + **elliptic regularity** (Ω suff. smooth) give

$$|\chi|_{H^1(0,T;H^1(\Gamma_c)) \cap L^\infty(0,T;H^2(\Omega))} \leq C$$
$$|\partial I_{[0,1]}(\chi)|_{L^\infty(0,T;L^2(\Gamma_c))} \leq C$$

& by comparison

$$|\partial I_{(-\infty,0]}(\dot{\chi})|_{L^\infty(0,T;L^2(\Gamma_c))} \leq C$$
$$|\chi|_{W^{1,\infty}(0,T;L^2(\Gamma_c))} \leq C$$

Third a priori estimate

- By comparison in the first equation

$$|\partial I_{(-\infty,0]}(u_N)\mathbf{n} + |\mathcal{R}(-\eta)|\mathbf{d}(\dot{\mathbf{u}}_T)|_{L^2(0,T;H^{-1/2}(\Gamma_c))} \leq C$$

Third a priori estimate

- By comparison in the first equation

$$|\partial I_{(-\infty,0]}(u_N)\mathbf{n} + |\mathcal{R}(-\eta)|\mathbf{d}(\dot{\mathbf{u}}_T)|_{L^2(0,T;H^{-1/2}(\Gamma_c))} \leq C$$

$|\mathcal{R}(-\eta)|\mathbf{d}(\dot{\mathbf{u}}_T)$ & $\partial I_{(-\infty,0]}(u_N)\mathbf{n}$ are **orthogonal**, hence

$$\begin{cases} |\partial I_{(-\infty,0]}(u_N)\mathbf{n}|_{L^2(0,T;H^{-1/2}(\Gamma_c))} \leq C, \\ ||\mathcal{R}(-\eta)|\mathbf{d}(\dot{\mathbf{u}}_T)|_{L^2(0,T;H^{-1/2}(\Gamma_c))} \leq C \end{cases}$$

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- In addition (from its definition), $|\mathbf{d}(\dot{\mathbf{u}}_T)|_{L^\infty((0,T)\times\Gamma_c)} \leq 1$

Passage to the limit

- by compactness and monotonicity-semicontinuity arguments
- identification of weak limits for maximal monotone operators
  - semicontinuity arguments and weak/strong convergence for $\partial I_{[0,1]}(\chi)$ and $\partial I_{(-\infty,0]}(\dot{\chi})$
  - ....
  - **Main difficulty**: the terms

$$|\mathcal{R}(-\eta)|\mathbf{d}(\dot{\mathbf{u}}_T) \qquad \& \qquad \eta \in \partial I_{(-\infty,0]}(u_N)$$

simultaneously present in the first equation.

Identification of the nonlinearities

- First step: identification of $\partial I_{(-\infty,0]}(u_N)$

  $\rightsquigarrow$ by semicontinuity, passing to the limit weakly in the first equation

- Second step: identification of $|\mathcal{R}(-\eta)|\mathbf{d}(\dot{\mathbf{u}}_T)$

  $\rightsquigarrow$ by compactifying character of $\mathcal{R}$

  - $\mathcal{R} : L^2(0,T;H^{-1/2}(\Gamma_c)) \to L^2(0,T;L^2(\Gamma_c))$
  - for all $\eta_\varepsilon, \eta \in L^2(0,T;H^{-1/2}(\Gamma_c))$

$$\eta_\varepsilon \rightharpoonup \eta \text{ **weakly** in } L^2(0,T;H^{-1/2}(\Gamma_c))$$

$$\Rightarrow \mathcal{R}(\eta_\varepsilon) \to \mathcal{R}(\eta) \text{ **strongly** in } L^2(0,T;L^2(\Gamma_c))$$

The nonisothermal case

To take into account **thermal effects:**

- in the **bulk domain** Ω:
  - $\varepsilon(\mathbf{u})$
  - θ **(volume absolute temperature)**
- on the **contact surface** Γ_c:
  - χ
  - θ_s (surface absolute temperature)

The nonisothermal case

To take into account **thermal effects:**

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  - $\varepsilon(\mathbf{u})$
  - θ **(volume absolute temperature)**
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  - χ
  - θ_s (surface absolute temperature)

- **friction** *coefficient* **depends on the thermal gap** $(\theta_{|\Gamma_c} - \theta_s)$
  - $$\nu \rightsquigarrow \nu(\theta_{|\Gamma_c} - \theta_s)$$
  - **contributions due to friction as source of heat on** Γ_c **(***heat generated by friction***)**.

The equations for θ and θ_{s}

Entropy balance equations (rescaled energy balance, under **small perturbation assumption)**

- **on the bulk domain:**

$$\begin{cases} \partial_t s + \operatorname{div} \mathbf{Q} = h & \text{in } \Omega \times (0,T), \\ \begin{cases} \mathbf{Q} \cdot \mathbf{n} = F & \text{in } \Gamma_{\mathrm{c}} \times (0,T), \\ \mathbf{Q} \cdot \mathbf{n} = 0 & \text{in } \partial\Omega \setminus \Gamma_c \times (0,T), \end{cases} \end{cases}$$

$$\begin{cases} s \text{ volume entropy} \\ \mathbf{Q} \text{ volume entropy flux} \\ F \text{ entropy exchanged through } \Gamma_c \\ h \text{ external source} \end{cases}$$

- **on the contact surface:**

$$\begin{cases} \partial_t s_s + \operatorname{div}_{\mathrm{s}} \mathbf{Q}_s = F & \text{in } \Gamma_{\mathrm{c}} \times (0,T), \\ \mathbf{Q}_s \cdot \mathbf{n}_s = 0 & \text{on } \partial\Gamma_{\mathrm{c}} \times (0,T), \end{cases}$$

$$\begin{cases} s_s \text{ surface entropy} \\ \mathbf{Q}_s \text{ surface entropy flux} \\ F \text{ entropy exchanged through } \Gamma_c \end{cases}$$

♠ **Entropy balance**: see **[Bonetti-Colli-Fabrizio-Gilardi '08]**, also **[Bonetti-Colli-Frémond '03, Bonetti-B-Rossi '09.]**

Entropy equation for θ on the bulk domain

$$\partial_t(\log\theta) - \operatorname{div}\dot{\mathbf{u}} - \Delta\theta = h \quad \text{on } \Omega\times(0,T),$$

$$\partial_n\theta = \begin{cases} 0 & \text{on } \partial\Omega\setminus\Gamma_c\times(0,T),\\ -\chi(\theta-\theta_s)-\nu'(\theta-\theta_s)|\mathcal{R}(-\partial I_{]-\infty,0]}(u_N))||\dot{\mathbf{u}}_T| & \text{on } \Gamma_c\times(0,T),\end{cases}$$

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Entropy equation for θ_s on the contact surface

$$\begin{aligned}\partial_t(\log\theta_s) - \lambda'(\chi)\dot{\chi} - \Delta_s\theta_s &=\\ &= \chi(\theta-\theta_s) + \nu'(\theta-\theta_s)|\mathcal{R}(-\partial I_{]-\infty,0]}(u_N))||\dot{\mathbf{u}}_T| \quad \text{in } \Gamma_c\times(0,T)\end{aligned}$$

$$\partial_n\theta_s = 0 \quad \text{on } \partial\Gamma_c\times(0,T)\,.$$

Entropy equation for θ on the bulk domain

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Entropy equation for θ_s on the contact surface

$$\begin{aligned} &\partial_t(\log\theta_s) - \lambda'(\chi)\dot{\chi} - \Delta_s\theta_s = \\ &\qquad = \chi(\theta-\theta_s) + \nu'(\theta-\theta_s)|\mathcal{R}(-\partial I_{]-\infty,0]}(u_N))||\dot{\mathbf{u}}_T| \quad \text{in } \Gamma_c \times (0,T) \\ &\partial_n\theta_s = 0 \quad \text{on } \partial\Gamma_c \times (0,T)\,. \end{aligned}$$

♣ we deduce directly $\theta, \theta_s > 0$, crucial for thermodynamical consistency

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♣ we deduce directly $\theta, \theta_s > 0$, crucial for thermodynamical consistency

♠ **singular character of the θ, θ_s-equations** (θ-**equation is coupled with a third type boundary condition**).

The full system

$$-\operatorname{div}\left(K\varepsilon(\mathbf{u})+K_v\varepsilon(\dot{\mathbf{u}})+\theta\mathbf{1}\right)=\mathbf{f}\quad\text{in }\Omega\times(0,T),$$

$$\mathbf{u}=\mathbf{0}\quad\text{on }\Gamma_1\times(0,T),\quad (K\varepsilon(\mathbf{u})+K_v\varepsilon(\dot{\mathbf{u}})+\theta\mathbf{1})\mathbf{n}=\mathbf{g}\quad\text{on }\Gamma_2\times(0,T),$$

$$(K\varepsilon(\mathbf{u})+K_v\varepsilon(\dot{\mathbf{u}})+\theta\mathbf{1})\mathbf{n}+\chi\mathbf{u}+\partial I_{]-\infty,0]}(u_N)\mathbf{n}+\nu(\theta-\theta_s)|\mathcal{R}(-\partial I_{]-\infty,0]}(u_N))|\mathbf{d}(\dot{\mathbf{u}}_T)\ni\mathbf{0}$$

$$\dot\chi-\Delta_s\chi+\partial I_{[0,1]}(\chi)\ni\omega-\lambda'(\chi)(\theta_s)-\frac{1}{2}|\mathbf{u}|^2\quad\text{in }\Gamma_c\times(0,T),$$

$$\partial_n\chi=0\quad\text{on }\partial\Gamma_c\times(0,T)$$

$$\partial_t(\log\theta)-\operatorname{div}\mathbf{u}_t-\Delta\theta=h\quad\text{in }\Omega\times(0,T),$$

$$\partial_n\theta=\begin{cases}0 & \text{on }\partial\Omega\setminus\Gamma_c\times(0,T),\\ -\chi(\theta-\theta_s)-\nu'(\theta-\theta_s)|\mathcal{R}(-\partial I_{]-\infty,0]}(u_N))||\dot{\mathbf{u}}_T| & \text{on }\Gamma_c\times(0,T),\end{cases}$$

$$\partial_t(\log\theta_s)-\lambda'(\chi)\chi_t-\Delta_s\theta_s=\chi(\theta-\theta_s)+\nu'(\theta-\theta_s)|\mathcal{R}(-\partial I_{]-\infty,0]}(u_N))||\dot{\mathbf{u}}_T|\quad\text{in }\Gamma_c\times(0,T),$$

$$\partial_n\theta_s=0\quad\text{on }\partial\Gamma_c\times(0,T)\qquad+\text{ Cauchy conditions}$$

$$-\operatorname{div}\left(K\varepsilon(\mathbf{u})+K_v\varepsilon(\dot{\mathbf{u}})+\theta\mathbf{1}\right)=\mathbf{f} \quad \text{in } \Omega\times(0,T),$$

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$$(K\varepsilon(\mathbf{u})+K_v\varepsilon(\dot{\mathbf{u}})+\theta\mathbf{1})\mathbf{n}+\chi\mathbf{u}+\partial I_{]-\infty,0]}(u_N)\mathbf{n}+\boxed{\nu(\theta-\theta_s)|\mathcal{R}(-\partial I_{]-\infty,0]}(u_N))|\mathbf{d}(\dot{\mathbf{u}}_T)}\ni\mathbf{0}$$

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♠ **Main difficulty:** **boundary coupling terms** (thermal & frictional effects)

$$-\operatorname{div}\left(K\varepsilon(\mathbf{u}) + K_v\varepsilon(\dot{\mathbf{u}}) + \theta\mathbf{1}\right) = \mathbf{f} \quad \text{in } \Omega\times(0,T),$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_1\times(0,T), \quad (K\varepsilon(\mathbf{u}) + K_v\varepsilon(\dot{\mathbf{u}}) + \theta\mathbf{1})\mathbf{n} = \mathbf{g} \quad \text{on } \Gamma_2\times(0,T),$$

$$(K\varepsilon(\mathbf{u}) + K_v\varepsilon(\dot{\mathbf{u}}) + \theta\mathbf{1})\mathbf{n} + \chi\mathbf{u} + \partial I_{]-\infty,0]}(u_N)\mathbf{n} + \boxed{\nu(\theta-\theta_s)|\mathcal{R}(-\partial I_{]-\infty,0]}(u_N))|\mathbf{d}(\dot{\mathbf{u}}_T)} \ni \mathbf{0}$$

$$\dot{\chi} - \Delta_s\chi + \partial I_{[0,1]}(\chi) \ni \omega - \lambda'(\chi)(\theta_s) - \frac{1}{2}|\mathbf{u}|^2 \quad \text{in } \Gamma_c\times(0,T),$$

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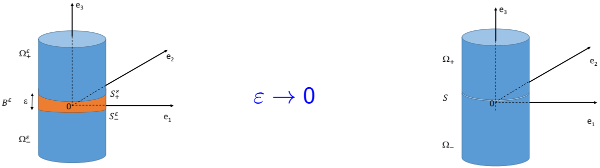
♠ **Main difficulty:** **boundary coupling terms** (thermal & frictional effects)

$\Rightarrow$ we need (...in addition...) **sufficient regularity** on θ and $\dot{\mathbf{u}}$ to control their traces $\rightsquigarrow$ suitable assumpt. on $\mathcal{R}$ and ν + careful estimates $\rightsquigarrow$ **Existence result for the full system**

An alternative approach: from volume to surface damage

⇝ **from volume damage to adhesive contact via dimensional reduction**

⇝ to recover the behaviour on **the interface S as limit of a thin medium** which links the body and the support (or two bodies) and which is ruled by its own evolution law



- from volume damage to adhesive contact via **asymptotic expansions method** **[Bonetti, B., Lebon, Rizzoni '17, Bonetti, B., Lebon '18]**
- from volume damage to delamination/adhesive contact **via variational techniques** **[Freddi, Paroni, Roubiček, Zanini '11, Mielke, Roubiček, Thomas '12]**

Outlook to the stochastic framework

To take into account

- unknown distribution of cracks and defects in the material
- fluctuations/phase changes at the microscopic level

$$\Downarrow$$

stochastic models of damage

Outlook to the stochastic framework

$$b(\partial_t \mathbf{u}, \mathbf{v}) + a(\mathbf{u}, \mathbf{v}) + \int_{\Gamma_c} \chi \mathbf{u} \cdot \mathbf{v} + \int_{\Gamma_c} \eta \mathbf{v} \cdot \mathbf{n} = \langle \mathbf{F}, \mathbf{v} \rangle \quad \forall \mathbf{v} \in \mathbf{W} \ \text{ a.e. in } (0, T)$$

$$\eta \in \partial I_{(-\infty,0]}(u_N) \quad \text{on } \Gamma_c \times (0, T)$$

$$\partial_t \chi - \Delta_s \chi + \partial I_{[0,1]}(\chi) \ni \omega - \frac{1}{2}|\mathbf{u}|^2 \quad \text{on } \Gamma_c \times (0, T),$$

$$\partial_{\mathbf{n}_s} \chi = 0 \quad \text{on } \partial\Gamma_c \times (0, T)$$

$$+ \ \textit{Cauchy} \quad \textit{conditions}$$

Outlook to the stochastic framework

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$$\boxed{\partial_t \chi - \Delta_s \chi + \partial I_{[0,1]}(\chi) \ni \omega - \frac{1}{2}|\mathbf{u}|^2 \quad \text{on } \Gamma_c \times (0, T),}$$

$$\partial_{\mathbf{n}_s} \chi = 0 \quad \text{on } \partial\Gamma_c \times (0, T)$$

$+$ *Cauchy conditions*

A stochastic model of damage [Bauzet, Bonetti, B., Lebon, Vallet, '17]

Stochastic Allen-Cahn equation with constraint

$$
\begin{cases}
\partial_t\left(\chi - \int_0^t h(\chi)dW\right) - \Delta\chi + \partial I_{[0,1]}(\chi) & \ni & w_s(\chi) + f & \text{in } \Omega \times D \times (0,T), \\
\partial_{\mathbf{n}}\chi & = & 0 & \text{in } \Omega \times \partial D \times (0,T), \\
\chi(\omega, x, t=0) & = & \chi_0(x) & \omega \in \Omega, x \in D.
\end{cases}
$$

- $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space, $D \subset \mathbb{R}^d$, $d \geqslant 1$
- χ the damage parameter, $0 \leqslant \chi \leqslant 1$
- $w_s : \mathbb{R} \to [0, +\infty[$ a Lipschitz-continuous function
- $f : \Omega \times D \times (0,T) \to \mathbb{R}$ a stochastic process
- $h : \mathbb{R} \to \mathbb{R}$ a Lipschitz-continuous function
- $W = (W_t)_{0 \leqslant t \leqslant T}$ a one dimensional Brownian motion defined on $(\Omega, \mathcal{F}, \mathbb{P})$.
- $\chi_0 : D \to \mathbb{R}$ the initial condition

A stochastic model of damage **[Bauzet, Bonetti, B., Lebon, Vallet, '17]**

Stochastic Allen-Cahn equation with constraint

$$
\begin{cases}
\partial_t\left(\chi - \int_0^t h(\chi)dW\right) - \Delta\chi + \partial I_{[0,1]}(\chi) \ni w_s(\chi) + f & \text{in } \Omega \times D \times (0,T), \\
\partial_{\mathbf{n}}\chi = 0 & \text{in } \Omega \times \partial D \times (0,T), \\
\chi(\omega, x, t=0) = \chi_0(x) & \omega \in \Omega, x \in D.
\end{cases}
$$

- Moreau-Yosida regularization of $\partial I_{[0,1]}(\cdot)$
- Existence and uniqueness for the time discretized system
- Uniforme estimates/passage to the limit procedure

$\rightsquigarrow$ **Existence and uniqueness result**