Convergence of monotone finite-volume schemes for hyperbolic scalar conservation laws with a stochastic force

Caroline Bauzet

Laboratoire de Mécanique et d'Acoustique

Joint work with
Vincent Castel, **Julia Charrier** and **Thierry Gallouët**

Aix Marseille université
cnrs
Centrale Marseille
INSTITUT de MATHÉMATIQUES de MARSEILLE

The Cauchy problem for a first-order scalar conservation law

$$\begin{cases} \partial_t u + \mathrm{div}_x \left[\vec{f}(.,.,u)\right] = 0 & \text{in } \mathbb{R}^d \times (0,T), \\ u(x, t=0) = u_0(x), & x \in \mathbb{R}^d. \end{cases}$$

Assumptions :

- $\vec{f} \in \mathcal{C}^1(\mathbb{R}^d \times [0,T] \times \mathbb{R}, \mathbb{R}^d)$ and $\forall (x,t,v)$, $\mathrm{div}_x \left[\vec{f}(x,t,v)\right] = 0$.
- $|\frac{\partial \vec{f}}{\partial v}(x,t,v)| \leqslant C_{\vec{f}}$ and $\frac{\partial \vec{f}}{\partial v}$ is locally Lipschitz w.r.t x, uniformly w.r.t (t,v).
- $u_0 \in L^\infty(\mathbb{R}^d)$.

Well known results :

- Existence of solutions by **parabolic** regularization : $\epsilon > 0$

$$\partial_t u_\epsilon + \mathrm{div}_x \left[\vec{f}(.,.,u_\epsilon)\right] - \epsilon \Delta u_\epsilon = 0 \text{ and } u_\epsilon(., t=0) = u_\epsilon^0(.).$$

- Uniqueness : concept of **entropy solution**.
- Numerical approximation : **finite-volume** method.

A stochastic PDE

$$\begin{cases} \partial_t \left(u - \int_0^{\cdot} g(u)dW \right) + \mathrm{div}_x[\vec{f}(.,.,u)] &= 0 \quad \text{in } \Omega \times \mathbb{R}^d \times (0,T), \\ u(\omega, x, t=0) &= u_0(x), \ \omega \in \Omega, x \in \mathbb{R}^d. \end{cases}$$

- $(\Omega, \mathcal{A}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0}, (W_t)_{t\geq 0})$ a stochastic basis with a standard real-valued **Brownian motion** $W = (W_t)_{t\geq 0}$.
- $g : \mathbb{R} \to \mathbb{R}$ a Lipschitz-continuous **bounded** function with $g(0) = 0$.
- $\int_0^{\cdot} g(u)dW$ the Itô integral of $g(u)$ (multiplicative noise).
- $\vec{f}(x, t, 0) = \vec{0}, \ \forall (x, t) \in \mathbb{R}^d \times [0, T]$ and same assumptions as previously.
- $\vec{f}$ is locally Lipschitz w.r.t x, uniformly w.r.t (t, v).
- $\vec{f}$ is locally Lipschitz w.r.t t, uniformly w.r.t (x, v).
- $u_0 \in L^2(\mathbb{R}^d)$.

State of the art

- **Theoretical studies**
  - **« Strong » entropy approach** : Feng & Nualart ('08), Chen, Ding & Karlsen ('12), Biswas & Majee ('14).
  - **Kinetic approach** : Debussche & Vovelle ('10), Hofmanová ('14), Kobayasi & Noboriguchi ('16).
  - **Entropy approach** : Bauzet, Vallet & Wittbold ('12, '14).
- **Numerical analysis**
  - **Time-splitting operator method** ($d = 1$ and $d \geqslant 1$) : Holden & Risebro ('91), Bauzet ('13), Karlsen & Storrøsten ('18).
  - **Semi-discrete Finite Volume Scheme** ($d = 1$) : Kröker & Rohde ('12), Koley, Majee & Vallet ('16).
  - **Flux-splitting FVS, monotone FVS** ($d \geqslant 1$), entropy approach : Bauzet, Charrier & Gallouët ('16, '17), Funaki, Gao & Hilhorst ('18).
  - **Monotone FVS** ($d \geqslant 1$), kinetic approach : Dotti & Vovelle ('18, '20).

References of the talk

- *Convergence of flux-splitting finite-volume schemes for hyperbolic scalar conservation laws with a multiplicative stochastic perturbation*, C. Bauzet, J. Charrier and T. Gallouët, *Math. of Comp.*, 2016.
  $\vec{f}(x,t,u) = \vec{v} f(u)$ with $\vec{v} \in \mathbb{R}^d$ and $f : \mathbb{R} \to \mathbb{R}$ Lipschitz-continuous.

- *Convergence of monotone finite-volume schemes for hyperbolic scalar conservation laws with a multiplicative noise*, C. Bauzet, J. Charrier and T. Gallouët, *SPDE : Analysis and Computations*, 2016.
  $\vec{f}(x,t,u) = \vec{v}(x,t) f(u)$ with $\vec{v} \in \mathcal{C}^1(\mathbb{R}^d \times [0,T], \mathbb{R}^d)$ and $f : \mathbb{R} \to \mathbb{R}$ Lipschitz-continuous.

- *Existence and uniqueness result for an hyperbolic scalar conservation law with a stochastic force using a finite-volume approximation*, C. Bauzet, V. Castel and J. Charrier, *JHDE*, 2020.
  $\vec{f}(x,t,u)$ (general case).

The brownian motion $W = (W_t)_{0 \leqslant t \leqslant T}$

- $W = (W_t)_{0 \leqslant t \leqslant T}$ is a stochastic process on $(\Omega, \mathcal{F}, \mathbb{P})$.
- $W_0 = 0$ (standard brownian motion).

$\forall s, t \in [0, T]$ with $t \geqslant s$

- $W_t - W_s \sim \mathcal{N}(0, t-s)$.
- $\mathbb{E}\left[W_t - W_s\right] = 0$.
- $\mathbb{E}\left[\left(W_t - W_s\right)^2\right] = t - s$.
- $(\mathcal{F}_t)_{0 \leqslant t \leqslant T}$ the filtration associated with $\mathcal{F}$ and generated by the brownian motion W ($\mathcal{F}_t$ contains the "story" of W up to time t).
- $W_t : \Omega \to \mathbb{R}$ is a random variable $\mathcal{F}_t$-measurable.
- If X is a random variable $\mathcal{F}_s$-measurable then $\mathbb{E}\left[(W_t - W_s)X\right] = 0$.

The Itô integral for a simple process

Set H a **Hilbert** space (for example $L^2(\mathbb{R}^d)$ or $H^1(\mathbb{R}^d)$).

Definition : simple process

$(\phi(t))_{0\leqslant t\leqslant T}$ is a simple process with values in H if there exist $0 = t_0 \leqslant \ldots \leqslant t_k \leqslant t_{k+1} = T$ and (k+1) random variables $\phi_0, \phi_1, \ldots, \phi_k : \Omega \to H$ such that $\forall n$, ϕ_n is $\mathcal{F}_{t_n}$-measurable and

$$\begin{array}{rcl} \phi(t) : \Omega & \to & H \\ \omega & \mapsto & \displaystyle\sum_{n=0}^{k} \phi_n 1_{[t_n, t_{n+1}[}(t). \end{array}$$

We denote by $S^2\big((0,T)\times\Omega; H\big)$ the set of simple processes with values in H.

The Itô integral of a simple process

$$\int_0^T \phi(s)dW(s) = \sum_{n=0}^{k} \int_{t_n}^{t_{n+1}} \phi(s)dW(s) = \sum_{n=0}^{k} \phi_n(W_{t_{n+1}} - W_{t_n}).$$

Properties of Itô integral

1. **Zero average :** $\mathbb{E}\left[\int_0^T \phi(s)\,\mathrm{d}W(s)\right] = 0.$
2. **Itô isometry :**
$$\mathbb{E}\left[\left\| \int_0^T \phi(s)\,\mathrm{d}W(s)\right\|_H^2\right] = \mathbb{E}\left[\int_0^T \|\phi(s)\|_H^2\,\mathrm{d}s\right].$$
3. **Linear continuity :** the application
$$\begin{array}{rcl} S^2\big((0,T)\times\Omega;H\big) & \to & C\big([0,T];L^2(\Omega;H)\big) \\ \phi & \mapsto & \displaystyle\int_0^{\cdot} \phi(s)dW(s) \text{ is linear and continuous.} \end{array}$$

Extension of the Itô integral

To the predictable processes $X \in \mathcal{N}_W^2(0,T;H) \subset L^2\big((0,T)\times\Omega;H\big)$ using the density of $S^2\big((0,T)\times\Omega;H\big)$ in $L^2\big((0,T)\times\Omega;H\big)$ with the norm $\mathbb{E}[\int_0^T \|.\|_X^2 ds]$.

Definition : stochastic entropy solution

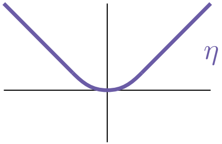
A process $u \in \mathcal{N}_w^2(0,T;L^2(\mathbb{R}^d)) \subset L^2((0,T)\times\Omega;L^2(\mathbb{R}^d))$ is a stochastic entropy solution if $u \in L^\infty(0,T;L^2(\Omega\times\mathbb{R}^d))$ and satisfies :

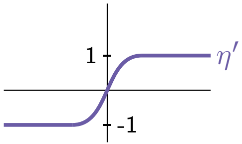
$$\begin{aligned}0 \quad \leq \quad & \int_{\mathbb{R}^d}\eta(u_0)\varphi(x,0)\,\mathrm{d}x + \int_0^T\int_{\mathbb{R}^d}\eta(u)\varphi_t(x,t)dxdt \\ & + \int_0^T\int_{\mathbb{R}^d}\left[\int_0^u \eta'(\sigma)\frac{\partial \vec{f}}{\partial v}(x,t,\sigma)d\sigma\right].\nabla_x\varphi(x,t)dxdt \\ & + \int_0^T\int_{\mathbb{R}^d}\eta'(u)g(u)\varphi(x,t)\,\mathrm{d}x\,\mathrm{d}W(t) \\ & + \frac{1}{2}\int_0^T\int_{\mathbb{R}^d}\eta''(u)g^2(u)\varphi(x,t)dxdt,\end{aligned}$$

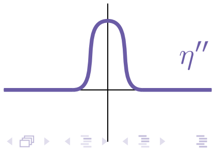
$\forall \eta \in \mathcal{C}^2(\mathbb{R})$ **convex**, $\forall \varphi \in \mathcal{C}_c^\infty(\mathbb{R}^d\times[0,T[)$, $\varphi \geqslant 0$, and $\mathbb{P}$-a.s. in Ω.

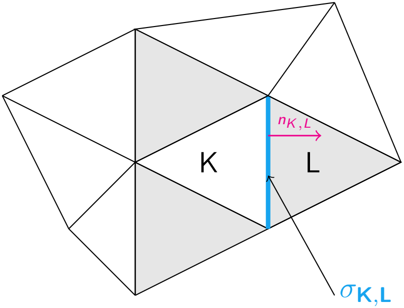
~~Kruzhkov's entropy~~

$\eta(u) = |u-\kappa|$, $\kappa \in \mathbb{R}$.

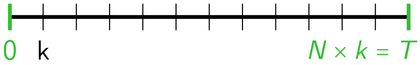








Uniform subdivision of $[0,T]$, $k>0$



Notations

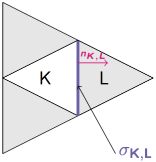
- $k=T/N$ the time step, $N\in\mathbb{N}^\star$.
- $\mathcal{T}$ a mesh of $\mathbb{R}^d$ and $\mathbf{h}=\text{size}(\mathcal{T})=\sup\{\text{diam}(K), K\in\mathcal{T}\}<\infty$.
- $\mathcal{N}(K)$ the set of control volumes neighbors of $K\in\mathcal{T}$.
- $\mathcal{E}_K$ the set of interfaces of the control volume K.
- $\sigma_{\mathbf{K,L}}$ the common interface between K and L, $K\in\mathcal{T}$, $L\in\mathcal{N}(K)$.
- $n_{K,L}$ the unit normal vector to interface $\sigma_{K,L}$, oriented from K to L.

A finite-volume scheme

$$\partial_t\left(u - \int_0^{\cdot} g(u)dW\right) + \mathrm{div}_x[\vec{f}(.,.,u)] = 0.$$

We approximate u by a « **piecewise constant** » function $u_{\mathcal{T},k} : \Omega \times \mathbb{R}^d \times (0,T) \to \mathbb{R}$

$$\boxed{u_{\mathcal{T},k}(\omega,x,t) = u_K^n, \quad \omega \in \Omega, x \in K, t \in [nk,(n+1)k)}$$



$$\begin{cases} u_K^0 &= \dfrac{1}{|K|}\displaystyle\int_K u_0(x)dx, \\ u_K^{n+1} - u_K^n + \dfrac{k}{|K|}\displaystyle\sum_{L\in\mathcal{N}(K)} Q_{K\to L}^n &= g(u_K^n)\Big(W((n+1)k) - W(nk)\Big), \end{cases}$$

where $Q_{K\to L}^n$ denotes the flow taken out of K towards L on $(nk,(n+1)k)$.

Choice of $Q^n_{K\to L}$

Condition : principle of conservation (mass, energy...) :

$$Q^n_{K\to L} = -Q^n_{L\to K}, \forall K, L \in \mathcal{T}, \forall n \in \{0, \ldots, N-1\}.$$

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Example : « upwind » choice

If $\vec{f} = \vec{v} \times f$ with $\vec{v} \in \mathbb{R}^d$ and $f : \mathbb{R} \to \mathbb{R}$ Lipschitz and **non-decreasing** : upwind scheme (left-sided, follow the sense of the wind)

$$Q^n_{K\to L} = |\sigma_{K,L}|(\vec{v}.n_{K,L}) f(u^n_{\sigma_{K,L}})$$

where

$$u^n_{\sigma_{K,L}} = \begin{cases} u^n_K & \text{if} \quad (\vec{v}.n_{K,L}) \geq 0 \\ u^n_L & \text{if} \quad (\vec{v}.n_{K,L}) < 0 \end{cases}$$

and so

$$Q^n_{K\to L} = |\sigma_{K,L}|\Big\{(\vec{v}.n_{K,L})^+ f(u^n_K) - (\vec{v}.n_{K,L})^- f(u^n_L)\Big\}.$$

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Example : « flux-splitting » choice with $\vec{v} \in \mathbb{R}^d$

If $\vec{f} = \vec{v} \times f$ with $\vec{v} \in \mathbb{R}^d$ and $f : \mathbb{R} \to \mathbb{R}$ Lipschitz thus $f = f_1 + f_2$, with $f_1 \nearrow$ and $f_2 \searrow$

- f_1 : upwind scheme.
- f_2 : downwind scheme.

$$Q^n_{K\to L} = |\sigma_{K,L}|\Big\{(\vec{v}.n_{K,L})^+\Big(f_1(u^n_K) + f_2(u^n_L)\Big) - (\vec{v}.n_{K,L})^-\Big(f_1(u^n_L) + f_2(u^n_K)\Big)\Big\},$$

Modified Lax-Friedrichs : $f_1(x) = \dfrac{f(x)}{2} + Dx$, $f_2(x) = \dfrac{f(x)}{2} - Dx$, with $D \geqslant \dfrac{C_f}{2}$.

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- f_1 : upwind scheme.
- f_2 : downwind scheme.

$$Q^n_{K\to L} = |\sigma_{K,L}|\Big\{(\vec{v}.n_{K,L})^+\big(f_1(u^n_K) + f_2(u^n_L)\big) - (\vec{v}.n_{K,L})^-\big(f_1(u^n_L) + f_2(u^n_K)\big)\Big\},$$

Modified Lax-Friedrichs : $f_1(x) = \dfrac{f(x)}{2} + Dx$, $f_2(x) = \dfrac{f(x)}{2} - Dx$, with $D \geqslant \dfrac{C_f}{2}$.

Theorem [Bauzet-Charrier-Gallouët (2015)]

When $Q^n_{K\to L}$ is of « flux-splitting » type then $u_{\mathcal{T},k}$ converges to the unique stochastic entropy solution of the problem in $L^p_{\text{loc}}\big(\Omega \times \mathbb{R}^d \times (0,T)\big)$, $\forall p < 2$.

Choice of $Q^n_{K\to L}$

Definition

$(F^n_{K,L})^n_{K,L}$ is a family of monotone numerical fluxes if for any $n\in\mathbb{N}$, $K\in\mathcal{T}$ and $L\in\mathcal{N}(K)$, $F^n_{K,L}:\mathbb{R}^2\to\mathbb{R}$ and for any $(a,b)\in\mathbb{R}^2$

- **Monotony** : $a\mapsto F^n_{K,L}(a,b)$ is $\nearrow$ and $b\mapsto F^n_{K,L}(a,b)$ is $\searrow$.
- **Regularity** : $F^n_{K,L}$ is Lip-diag : $\exists F_1,F_2>0$ such that $\forall a,b\in\mathbb{R}$

  $$|F^n_{K,L}(b,a)-F^n_{K,L}(a,a)|\le F_1|b-a| \text{ and } |F^n_{K,L}(a,b)-F^n_{K,L}(a,a)|\le F_2|b-a|.$$

- **Conservativity** : $F^n_{K,L}(a,b)=-F^n_{L,K}(b,a)$.
- **Consistency** : $F^n_{K,L}(a,a)=\dfrac{1}{k|\sigma_{K,L}|}\displaystyle\int_{nk}^{(n+1)k}\int_{\sigma_{K,L}}\vec{f}(x,t,a).n_{K,L}\,d\gamma(x)dt.$

$$Q^n_{K\to L}=|\sigma_{K,L}|F^n_{K,L}(u^n_K,u^n_L).$$

If $\vec{f} = \vec{v} \times f$ with $\vec{v} \in \mathbb{R}^d$ and $f : \mathbb{R} \to \mathbb{R}$ Lipschitz and **non-decreasing** :

$$\begin{cases} u_{\mathcal{T},k}(.,x,t) = u_K^n, x \in K, t \in [nk,(n+1)k), \\ u_K^0 = \dfrac{1}{|K|}\displaystyle\int_K u_0(x)dx, \\ u_K^{n+1} - u_K^n + \dfrac{k}{|K|}\displaystyle\sum_{\sigma_{K,L}\in\mathcal{E}_K} |\sigma_{K,L}|(\vec{v}.n_{K,L})f(u_{\sigma_{K,L}}^n) = g(u_K^n)\Big(W\left((n+1)k\right) - W\left(nk\right)\Big) \end{cases}$$

1. Stability estimates on $(u_{\mathcal{T},k})$.
2. Entropy estimates satisfied by $(u_{\mathcal{T},k})$.
3. Passage to the limit in the entropy estimates satisfied by $(u_{\mathcal{T},k})$.
4. Existence and uniqueness of the stochastic entropy solution u.

Stability estimates

Non-degeneracy condition on the spatial mesh :

$$\exists \bar{\alpha} > 0 / \forall K \in \mathcal{T} : \bar{\alpha} h^d \leq |K| \quad \text{and} \quad |\partial K| \leq \frac{1}{\bar{\alpha}} h^{d-1}.$$

Proposition : $L^\infty_t L^2_{\omega,x}$ estimate

Under the CFL condition : $k \leqslant \dfrac{\bar{\alpha}^2 h}{C_f \|\vec{v}\|_\infty}$, we have :

$$\forall n \in \{0, .., N-1\}, \quad \sum_{K \in \mathcal{T}} |K| \mathbb{E}\left[(u_K^n)^2 \right] \leqslant e^{T C_g^2} \|u_0\|^2_{L^2(\mathbb{R}^d)}.$$

Thus $(u_{\mathcal{T},k})$ is bounded in $L^\infty\left(0, T; L^2(\Omega \times \mathbb{R}^d)\right)$.

(where $\mathbb{E}[(u_K^n)^2] = \int_\Omega (u_K^n)^2 d\mathbb{P}$ and C_f, C_g are the Lipschitz constant of f and g).

Idea of the proof : multiply the FV scheme by u_K^n

CFL condition + monotony of f + properties of the Brownian motion.

Weak BV estimate

Let $R > h$. Under the CFL Condition

$$k \leqslant (1-\xi)\frac{\bar{\alpha}^2 h}{C_f \|\vec{v}\|_\infty}, \text{ for some } \xi \in (0,1),$$

There exists $C \in \mathbb{R}_+^\star$, only depending on $R, d, T, \bar{\alpha}, u_0, \vec{v}, \xi, C_f$ and C_g such that

$$\sum_{n=0}^{N-1} k \sum_{(K,L)\in\mathfrak{T}_n^R} |\sigma_{K,L}||\vec{v}.n_{K,L}|\mathbb{E}\Big[|f(u_K^n) - f(u_L^n)|\Big] \leqslant Ch^{-1/2},$$

where $\mathfrak{T}_n^R = \big\{(K,L) \in \mathcal{T}^2 \text{ such that } L \in \mathcal{N}(K), K, L \subset B(0,R) \text{ and } u_K^n > u_L^n\big\}$.

Usefulness

Control the difference between the numerical entropy flux and the entropy flux at $u_{\mathcal{T},k}$, namely $\int_0^{u_{\mathcal{T},k}} \eta'(\sigma) f'(\sigma) d\sigma$.

Our goal : entropy estimates satisfied by $(u_{\mathcal{T},k})$

For all convex functions $\eta \in \mathcal{C}^2(\mathbb{R})$, for all $\varphi \in \mathcal{C}_c^\infty\big(\mathbb{R}^d \times [0,T)\big)$ with $\varphi \geqslant 0$ and for all $\mathbb{P}$-measurable set A :

$$
\begin{aligned}
& \mathbb{E}\Big[1_A \int_{\mathbb{R}^d} \eta(u_0)\varphi(x,0)dx\Big] + \mathbb{E}\Big[1_A \int_0^T \int_{\mathbb{R}^d} \eta(u_{\mathcal{T},k})\varphi_t(x,t)dxdt\Big] \\
& + \mathbb{E}\Big[1_A \int_0^T \int_{\mathbb{R}^d} \Big(\int_0^{u_{\mathcal{T},k}} \eta'(\sigma)f'(\sigma)d\sigma\Big)\vec{v}.\nabla_x\varphi(x,t)dxdt\Big] \\
& + \mathbb{E}\Big[1_A \int_0^T \int_{\mathbb{R}^d} \eta'(u_{\mathcal{T},k})g(u_{\mathcal{T},k})\varphi(x,t)dxdW(t)\Big] \\
& + \frac{1}{2}\mathbb{E}\Big[1_A \int_0^T \int_{\mathbb{R}^d} \eta''(u_{\mathcal{T},k})g^2(u_{\mathcal{T},k})\varphi(x,t)dxdt\Big] \\
\geqslant\ & \mathbb{E}\Big[1_A R^{h,k}\Big],
\end{aligned}
$$

where for all $\mathbb{P}$-measurable set A, $\mathbb{E}\big[1_A R^{h,k}\big] \to 0$ as $k, h \to 0$.

Question : how to get these inequalities ?

In the deterministic case ($g = 0$)...

Using Kruzhkov entropies $\eta_\kappa(u) = |u - \kappa|$, $\kappa \in \mathbb{R}$.
Problem : Kruzhkov entropies are forbidden in the stochastic case !

In the deterministic case ($g=0$)...

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Problem : Kruzhkov entropies are forbidden in the stochastic case !

A time continuous approximation

$$\overline{u}_{\mathcal{T},k}(\omega,x,t)=\overline{u}_K^n(\omega,t),\ \omega\in\Omega,\ x\in K,\ t\in[nk,(n+1)k]$$

$$\overline{u}_K^n(s)=u_K^n-\frac{s-nk}{|K|}\underbrace{\sum_{\sigma_{K,L}\in\mathcal{E}_K}|\sigma_{K,L}|(\vec{v}.n_{K,L})f(u^n_{\sigma_{K,L}})}_{=\sum_{\sigma_{K,L}\in\mathcal{E}_K}|\sigma_{K,L}|(\vec{v}.n_{K,L})^-\left(f(u^n_{\sigma_{K,L}})-f(u^n_K)\right)}+\int_{nk}^s g(u_K^n)dW(t)$$

Thus $\overline{u}_K^n(nk)=u_K^n\ \ \forall n\in\{0,\dots,N\}$.

Convergence result

$$||u_{\mathcal{T},k}-\overline{u}_{\mathcal{T},k}||_{L^2(\Omega\times\mathbb{R}^d\times(0,T))}\to 0 \text{ as } h,k\to 0.$$

Set a convex function $\eta \in \mathcal{C}^2(\mathbb{R})$. Apply Itô's formula on $[nk,(n+1)k]$ to $\overline{u}_{\mathcal{T},k}$ and the functional $\Psi : (t,u) \mapsto \eta(u)$ to get $\mathbb{P}$ a-s. in Ω :

$$
\begin{aligned}
& -\eta\left(\overline{u}_K^n((n+1)k\right) - \eta\left(\overline{u}_K^n(nk)\right) \\
& + \frac{1}{|K|}\int_{nk}^{(n+1)k} \eta'(\overline{u}_K^n(t)) \sum_{\sigma_{K,L}\in\mathcal{E}_K} |\sigma_{K,L}|(\vec{v}.n_{K,L})^{-}\Big(f(u_{\sigma_{K,L}}^n) - f(u_K^n)\Big)dt \\
& + \int_{nk}^{(n+1)k} \eta'(\overline{u}_K^n(t))g(u_K^n)dW(t) \\
& + \frac{1}{2}\int_{nk}^{(n+1)k} \eta''(\overline{u}_K^n(t))g^2(u_K^n)dt \\
= {} & 0.
\end{aligned}
$$

For any convex function $\eta \in \mathcal{C}^2(\mathbb{R})$, any $\varphi \in \mathcal{C}_c^\infty\big(\mathbb{R}^d \times [0,T)\big)$ with $\varphi \geqslant 0$ and any $\mathbb{P}$-measurable set A :

$$
\begin{aligned}
&- \mathbb{E}\left[1_A \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_R} \big(\eta(u_K^{n+1}) - \eta(u_K^n)\big)|K|\varphi_K^n\right] \\
&+\mathbb{E}\left[1_A \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_R} \int_{nk}^{(n+1)k} \eta'(\bar{u}_{\mathcal{T},k}(t)) \sum_{\sigma_{K,L} \in \mathcal{E}_K} |\sigma_{K,L}|(\vec{v}.n_{K,L})^- \big(f(u_{\sigma_{K,L}}^n) - f(u_K^n)\big) dt \varphi_K^n\right] \\
&+ \mathbb{E}\left[1_A \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_R} \int_{nk}^{(n+1)k} \eta'(\bar{u}_{\mathcal{T},k}(t)) g(u_K^n) dW(t) |K|\varphi_K^n\right] \\
&+ \mathbb{E}\left[1_A \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_R} \frac{1}{2} \int_{nk}^{(n+1)k} \eta''(\bar{u}_{\mathcal{T},k}(t)) g^2(u_K^n) dt |K|\varphi_K^n\right] \\
&= 0.
\end{aligned}
$$

where $\varphi_K^n = \dfrac{1}{|K|}\displaystyle\int_K \varphi(x, nk)\,dx.$

$$\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{nk}^{(n+1)k}\eta'(\bar{u}_{\mathcal{T},k}(t))\sum_{\sigma_{K,L}\in\mathcal{E}_K}|\sigma_{K,L}|(\vec{v}.n_{K,L})^-\Big(f(u^n_{\sigma_{K,L}})-f(u^n_K)\Big)dt\varphi^n_K$$

$$\leqslant \int_0^T\int_{\mathbb{R}^d}F^\eta(u_{\mathcal{T},k})\vec{v}.\nabla_x\varphi(x,t)dxdt+\tilde{R}_2^{h,k}+R_2^{h,k},\ \text{with } \mathbb{E}\Big[1_A(\tilde{R}_2^{h,k}+R_2^{h,k})\Big]\xrightarrow[h\to 0]{} 0.$$

$\forall a\in\mathbb{R},\ F^\eta(a)=\int_0^a\eta'(\sigma)f'(\sigma)d\sigma$ denotes the entropy flux.

$$\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{nk}^{(n+1)k}\eta'(\bar{u}_{\mathcal{T},k}(t))\sum_{\sigma_{K,L}\in\mathcal{E}_K}|\sigma_{K,L}|(\vec{v}.n_{K,L})^-\Big(f(u^n_{\sigma_{K,L}})-f(u^n_K)\Big)\varphi^n_K dt$$

$$-\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{nk}^{(n+1)k}\eta'(u^n_K)\sum_{\sigma_{K,L}\in\mathcal{E}_K}|\sigma_{K,L}|(\vec{v}.n_{K,L})^-\Big(f(u^n_{\sigma_{K,L}})-f(u^n_K)\Big)\varphi^n_K dt$$

$$+\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{nk}^{(n+1)k}\eta'(u^n_K)\sum_{\sigma_{K,L}\in\mathcal{E}_K}|\sigma_{K,L}|(\vec{v}.n_{K,L})^-\Big(f(u^n_{\sigma_{K,L}})-f(u^n_K)\Big)\varphi^n_K dt$$

$$-\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{nk}^{(n+1)k}\sum_{\sigma_{K,L}\in\mathcal{E}_K}|\sigma_{K,L}|(\vec{v}.n_{K,L})^-\Big(F^\eta(u^n_{\sigma_{K,L}})-F^\eta(u^n_K)\Big)\varphi^n_K dt$$

$$+\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{nk}^{(n+1)k}\sum_{\sigma_{K,L}\in\mathcal{E}_K}|\sigma_{K,L}|(\vec{v}.n_{K,L})^-\Big(F^\eta(u^n_{\sigma_{K,L}})-F^\eta(u^n_K)\Big)\varphi^n_K dt$$

$$-\int_0^T\int_{\mathbb{R}^d}F^\eta(u_{\mathcal{T},k})\vec{v}.\nabla_x\varphi(x,t)dxdt+\int_0^T\int_{\mathbb{R}^d}F^\eta(u_{\mathcal{T},k})\vec{v}.\nabla_x\varphi(x,t)dxdt$$

$$\leqslant\int_0^T\int_{\mathbb{R}^d}F^\eta(u_{\mathcal{T},k})\vec{v}.\nabla_x\varphi(x,t)dxdt+\tilde{R}_2^{h,k}+R_2^{h,k},\ \text{ with } \mathbb{E}\Big[1_A(\tilde{R}_2^{h,k}+R_2^{h,k})\Big]\xrightarrow[h\to 0]{}0.$$

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$$-\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{nk}^{(n+1)k}\eta'(u^n_K)\sum_{\sigma_{K,L}\in\mathcal{E}_K}|\sigma_{K,L}|(\vec{v}.n_{K,L})^-\left(f(u^n_{\sigma_{K,L}})-f(u^n_K)\right)\varphi^n_K dt$$

$$+\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{nk}^{(n+1)k}\sum_{\sigma_{K,L}\in\mathcal{E}_K}|\sigma_{K,L}|(\vec{v}.n_{K,L})^-\eta'(u^n_K)\left(f(u^n_{\sigma_{K,L}})-f(u^n_K)\right)\varphi^n_K dt$$

$$-\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{nk}^{(n+1)k}\sum_{\sigma_{K,L}\in\mathcal{E}_K}|\sigma_{K,L}|(\vec{v}.n_{K,L})^-\left(F^\eta(u^n_{\sigma_{K,L}})-F^\eta(u^n_K)\right)\varphi^n_K dt$$

$$+\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{nk}^{(n+1)k}\sum_{\sigma_{K,L}\in\mathcal{E}_K}|\sigma_{K,L}|(\vec{v}.n_{K,L})^-\left(F^\eta(u^n_{\sigma_{K,L}})-F^\eta(u^n_K)\right)\varphi^n_K dt$$

$$-\int_0^T\int_{\mathbb{R}^d}F^\eta(u_{\mathcal{T},k})\vec{v}.\nabla_x\varphi(x,t)dxdt+\int_0^T\int_{\mathbb{R}^d}F^\eta(u_{\mathcal{T},k})\vec{v}.\nabla_x\varphi(x,t)dxdt$$

$$\leqslant\int_0^T\int_{\mathbb{R}^d}F^\eta(u_{\mathcal{T},k})\vec{v}.\nabla_x\varphi(x,t)dxdt+\tilde{R}_2^{h,k}+R_2^{h,k},\ \text{with } \mathbb{E}\left[1_A(\tilde{R}_2^{h,k}+R_2^{h,k})\right]\xrightarrow[h\to 0]{}0.$$

$\forall a\in\mathbb{R},\ F^\eta(a)=\int_0^a\eta'(\sigma)f'(\sigma)d\sigma$ denotes the entropy flux.

$$\mathbb{E}\Big[1_A\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{nk}^{(n+1)k}\eta'(\bar{u}_{\mathcal{T},k}(t))\sum_{\sigma_{K,L}\in\mathcal{E}_K}|\sigma_{K,L}|(\vec{v}.n_{K,L})^-\big(f(u^n_{\sigma_{K,L}})-f(u^n_K)\big)\varphi^n_K dt\Big]$$

$$-\mathbb{E}\Big[1_A\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{nk}^{(n+1)k}\eta'(u^n_K)\sum_{\sigma_{K,L}\in\mathcal{E}_K}|\sigma_{K,L}|(\vec{v}.n_{K,L})^-\big(f(u^n_{\sigma_{K,L}})-f(u^n_K)\big)\varphi^n_K dt\Big]$$

$$=\mathbb{E}\Big[1_A\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{nk}^{(n+1)k}\big[\eta'(\bar{u}_{\mathcal{T},k}(t))-\eta'(u^n_K)\big]dt\varphi^n_K$$

$$\times\sum_{\sigma_{K,L}\in\mathcal{E}_K}|\sigma_{K,L}|(\vec{v}.n_{K,L})^-\big(f(u^n_{\sigma_{K,L}})-f(u^n_K)\big)\Big]$$

$\rightarrow 0$ as $h\rightarrow 0$ and $\dfrac{k}{h}\rightarrow 0$.

$$\bar{u}_{\mathcal{T},k}(t)-u^n_K = -\frac{s-nk}{|K|}\sum_{\sigma_{K,L}\in\mathcal{E}_K}|\sigma_{K,L}|(\vec{v}.n_{K,L})^-\big(f(u^n_{\sigma_{K,L}})-f(u^n_K)\big)$$

$$+\int_{nk}^{s}g(u^n_K)dW(t),\ \ \forall t\in[nk,(n+1)k].$$

Question : how to show that

$$\mathbb{E}\Big[1_A \sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{nk}^{(n+1)k}\sum_{\sigma_{K,L}\in\mathcal{E}_K}|\sigma_{K,L}|(\vec{v}.n_{K,L})^-\eta'(u_K^n)\big(f(u_{\sigma_{K,L}}^n)-f(u_K^n)\big)\varphi_K^n dt\Big]$$

$$-\mathbb{E}\Big[1_A \sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{nk}^{(n+1)k}\sum_{\sigma_{K,L}\in\mathcal{E}_K}|\sigma_{K,L}|(\vec{v}.n_{K,L})^-\Big(\underbrace{F^\eta(u_{\sigma_{K,L}}^n)-F^\eta(u_K^n)}_{=\int_{u_K^n}^{u_{\sigma_{K,L}}^n}\eta'(s)f'(s)ds}\Big)\varphi_K^n dt\Big]$$

$\leqslant 0$?

Answer : since f and η' are non-decreasing, one gets :

$$\eta'(u_K^n)(f(u_{\sigma_{K,L}}^n)-f(u_K^n))-\big(F^\eta(u_{\sigma_{K,L}}^n)-F^\eta(u_K^n)\big)$$

$$=\int_{u_K^n}^{u_{\sigma_{K,L}}^n}\big(\eta'(u_K^n)-\eta'(s)\big)f'(s)ds$$

$$\leqslant 0.$$

Question : how to show that

$$\mathbb{E}\left[1_A \sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}_R} \int_{nk}^{(n+1)k} \sum_{\sigma_{K,L}\in\mathcal{E}_K} |\sigma_{K,L}|(\vec{v}.n_{K,L})^-\left(F^\eta(u^n_{\sigma_{K,L}}) - F^\eta(u^n_K)\right)\varphi^n_K dt\right]$$

$$-\mathbb{E}\left[1_A \int_0^T \int_{\mathbb{R}^d} F^\eta(u_{\mathcal{T},k})\vec{v}.\nabla_x\varphi(x,t)dxdt\right] \to 0 \text{ as } h \to 0 \ ?$$

Answer : **Weak BV estimates** :

$$\sum_{n=0}^{N-1} k \sum_{(K,L)\in\mathfrak{T}^R_n} |\sigma_{K,L}||\vec{v}.n_{K,L}|\mathbb{E}\left[|f(u^n_K) - f(u^n_L)|\right] \leqslant Ch^{-1/2}.$$

$$\forall a \in \mathbb{R},\ F^\eta(a) = \int_0^a \eta'(\sigma)f'(\sigma)d\sigma$$

In brief

$$A^{h,k} = \mathbb{E}\Big[1_A \sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}_R} \int_{nk}^{(n+1)k} \eta'(\bar{u}_{\mathcal{T},k}(t)) \sum_{\sigma_{K,L}\in\mathcal{E}_K} |\sigma_{K,L}|(\vec{v}.n_{K,L})^-\Big(f(u^n_{\sigma_{K,L}}) - f(u^n_K)\Big)dt\varphi^n_K\Big]$$

$$= \underbrace{A^{h,k} - B^{h,k}}_{\to 0} + \underbrace{B^{h,k} - C^{h,k}}_{\leqslant 0} + \underbrace{C^{h,k} - D^{h,k}}_{\to 0} + \mathbb{E}\Big[1_A \int_0^T \int_{\mathbb{R}^d} F^\eta(u_{\mathcal{T},k})\vec{v}.\nabla_x\varphi(x,t)dxdt\Big]$$

$$\leqslant \underbrace{\mathbb{E}\Big[1_A R^{h,k}\Big]}_{\to 0 \text{ as } h, \frac{k}{h}\to 0} + \mathbb{E}\Big[1_A \int_0^T \int_{\mathbb{R}^d} F^\eta(u_{\mathcal{T},k})\vec{v}.\nabla_x\varphi(x,t)dxdt\Big].$$

Key points :

- $A^{h,k} - B^{h,k} \to 0$: assume that $\dfrac{k}{h} \to 0$.
- $B^{h,k} - C^{h,k} \leqslant 0$: Monotony of f and η'.
- $C^{h,k} - D^{h,k} \to 0$: Weak BV estimate.

Extension to general monotone numerical fluxes

For a flux function $(x,t,u) \mapsto \vec{f}(x,t,u)$ with $Q^n_{K\to L} = |\sigma_{K,L}| F^n_{K,L}(u^n_K, u^n_L)$.

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Decomposition of $F^n_{K,L}$: [C. Chainais-Hillairet thesis]

There exists $\theta^n_{K,L} : \mathbb{R}^2 \to [0,1]$ such that $\forall a, b \in \mathbb{R}$

$$F^n_{K,L}(a,b) = \theta^n_{K,L}(a,b) \underbrace{F^{n,G}_{K,L}(a,b)}_{\text{Godunov}} + (1-\theta^n_{K,L}(a,b)) \underbrace{F^{n,LF}_{K,L}(a,b)}_{Lax-Friedrichs},$$

$$\text{where } F^{n,G}_{K,L}(a,b) = \begin{cases} \min\limits_{s\in[a,b]} f^n_{K,L}(s) & \text{if } a \leqslant b, \\ \max\limits_{s\in[b,a]} f^n_{K,L}(s) & \text{if } a \geqslant b. \end{cases}$$

$$F^{n,LF}_{K,L}(a,b) = \frac{f^n_{K,L}(a) + f^n_{K,L}(b)}{2} - D(b-a) \text{ with } D = \max(2F_1, 2F_2, C_{\vec{f}})$$

$$\text{and } f^n_{K,L}(s) = \frac{1}{k|\sigma_{K,L}|} \int_{nk}^{(n+1)k} \int_{\sigma_{K,L}} \vec{f}(x,t,s)\, d\gamma(x)\, dt.$$

Entropy inequalities satisfied by $u_{\mathcal{T},k}$

For any convex function $\eta \in \mathcal{C}^2(\mathbb{R})$, any $\varphi \in \mathcal{C}_c^\infty\big(\mathbb{R}^d \times [0,T)\big)$ with $\varphi \geqslant 0$ and for any $\mathbb{P}$-measurable set A :

$$
\begin{aligned}
&\mathbb{E}\Big[1_A \int_{\mathbb{R}^d} \eta(u_0)\varphi(x,0)\,dx + \mathbb{E}\Big[1_A \int_0^T \int_{\mathbb{R}^d} \eta(u_{\mathcal{T},k})\varphi_t(x,t)\,dxdt\Big] \\
&+\mathbb{E}\Big[1_A \int_0^T \int_{\mathbb{R}^d} F^\eta(u_{\mathcal{T},k})\vec{v}.\nabla_x\varphi(x,t)\,dxdt\Big] \\
&+\mathbb{E}\Big[1_A \int_0^T \int_{\mathbb{R}^d} \eta'(u_{\mathcal{T},k})g(u_{\mathcal{T},k})\varphi(x,t)\,dxdW(t)\Big] \\
&+\frac{1}{2}\mathbb{E}\Big[1_A \int_0^T \int_{\mathbb{R}^d} \eta''(u_{\mathcal{T},k})g^2(u_{\mathcal{T},k})\varphi(x,t)\,dxdt\Big] \\
\geqslant\ \ & \mathbb{E}\Big[1_A R^{h,k}\Big], \text{where } \mathbb{E}\Big[1_A R^{h,k}\Big] \to 0 \text{ as } h, \frac{k}{h} \to 0.
\end{aligned}
$$

Assumptions due to the noise : $\frac{k}{h} \to 0$ and g is bounded.

Compactness argument

Young measures, PDE and **Prohorov's** theorem

$(u_{\mathcal{T},k})_{\mathcal{T},k}$ is bounded in $L^2(\Omega \times Q)$ $\Rightarrow$ there exists a **Young measure**

$$\nu : \Omega \times Q \times \mathbb{R} \to \mathbb{R} \text{ such that : } \nu_{\mathcal{T},k} \xrightarrow[\text{measures}]{\text{Young}} \nu.$$

$Q = (0,T) \times \mathbb{R}^d$

- ⋆ $\psi : (\omega, x, t, \lambda) \in \Omega \times Q \times \mathbb{R} \mapsto \psi(\omega, x, t, \lambda) \in \mathbb{R}$ a **Carathéodory** function
- ⋆ $\psi(., u_{\mathcal{T},k})$ is **uniformly integrable**

$$\int_{\Omega \times Q \times \mathbb{R}} \psi \, d\nu_{\mathcal{T},k} \xrightarrow[h \to 0]{} \int_{\Omega \times Q \times \mathbb{R}} \psi \, d\nu$$

$$\begin{array}{ccc} \| & & \| \\ \displaystyle\int_{\Omega \times Q} \psi(., u_{\mathcal{T},k}(\omega, x, t)) \, dx dt d\mathbb{P} & & ? \end{array}$$

Compactness argument

Young measures, PDE and **Prohorov's** theorem

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- $\star$ $\psi : (\omega, x, t, \lambda) \in \Omega \times Q \times \mathbb{R} \mapsto \psi(\omega, x, t, \lambda) \in \mathbb{R}$ a **Carathéodory** function
- $\star$ $\psi(., u_{\mathcal{T},k})$ is **uniformly integrable**

$$\int_{\Omega \times Q \times \mathbb{R}} \psi \, d\nu_{\mathcal{T},k} \xrightarrow[h \to 0]{} \int_{\Omega \times Q \times \mathbb{R}} \psi \, d\nu$$

$$\int_{\Omega \times Q} \psi(., u_{\mathcal{T},k}(\omega, x, t)) \, dx dt dP \qquad \int_{\Omega \times Q} \int_0^1 \psi(., \mu(\omega, x, t, \alpha)) \, d\alpha dx dt d\mathbb{P}.$$

Notion of entropy process $\mu \in L^2(\Omega \times Q \times (0,1))$ **[Eymard-Gallouët-Herbin]**.

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For any convex function $\eta \in \mathcal{C}^2(\mathbb{R})$, any function $\varphi \in \mathcal{C}_c^\infty\left(\mathbb{R}^d \times [0,T)\right)$ with $\varphi \geqslant 0$ and for any $\mathbb{P}$-measurable set A :

$$
\begin{aligned}
&\mathbb{E}\left[1_A \int_{\mathbb{R}^d} \eta(u_0)\varphi(x,0)dx\right] + \mathbb{E}\left[\int_0^T \int_{\mathbb{R}^d} 1_A \eta(u_{\mathcal{T},k})\varphi_t(x,t)dxdt\right]\\
&+\mathbb{E}\left[\int_0^T \int_{\mathbb{R}^d} 1_A F^\eta(u_{\mathcal{T},k})\vec{v}.\nabla_x\varphi(x,t)dxdt\right]\\
&+\mathbb{E}\left[\int_0^T \int_{\mathbb{R}^d} 1_A \eta'(u_{\mathcal{T},k})g(u_{\mathcal{T},k})\varphi(x,t)dxdW(t)\right]\\
&+\frac{1}{2}\mathbb{E}\left[\int_0^T \int_{\mathbb{R}^d} 1_A \eta''(u_{\mathcal{T},k})g^2(u_{\mathcal{T},k})\varphi(x,t)dxdt\right]\\
\geqslant\ &\mathbb{E}\left[1_A R^{h,k}\right].
\end{aligned}
$$

Convergence to a measure valued-entropy solution

$(u_{\mathcal{T},k})$ converges in the sense of Young measures to a function μ belonging to $\mathcal{N}_w^2\Big(0,T,L^2(\mathbb{R}^d\times(0,1))\Big)\cap L^\infty\Big(0,T,L^2(\Omega\times\mathbb{R}^d\times(0,1))\Big)$ and satisfying

$$\mathbb{E}\left[1_A\int_{\mathbb{R}^d}\eta(u_0)\varphi(x,0)dx\right]+\mathbb{E}\left[1_A\int_0^T\int_{\mathbb{R}^d}\int_0^1\eta(\mu(.,\alpha))\varphi_t(x,t)\,d\alpha dxdt\right]$$
$$+\mathbb{E}\left[1_A\int_0^T\int_{\mathbb{R}^d}\int_0^1\Big(\int_0^{\mu(.,\alpha)}\eta'(\sigma)\frac{\partial\vec{f}}{\partial v}(x,t,\sigma)d\sigma\Big).\nabla_x\varphi(x,t)\,d\alpha dxdt\right]$$
$$+\mathbb{E}\left[1_A\int_0^T\int_{\mathbb{R}^d}\int_0^1\eta'(\mu(.,\alpha))g(\mu(.,\alpha))\varphi(x,t)\,d\alpha dxdW(t)\right]$$
$$+\frac{1}{2}\mathbb{E}\left[1_A\int_0^T\int_{\mathbb{R}^d}\int_0^1\eta''(\mu(.,\alpha))g^2(\mu(.,\alpha))\varphi(x,t)\,d\alpha dxdt\right]$$

$\geqslant\quad 0,\qquad$ for any P-measurable set A, any entropy η and any $\varphi\in\mathcal{C}_c^\infty\big(\mathbb{R}^d\times[0,T)\big)$.

The process μ is called a measure-valued entropy solution of our problem.

To conclude ...

- Stochastic version of Kruzkhov's doubling variables technique : comparing any measure valued solution $\tilde{\mu}$ with the parabolic u_ϵ or finite-volume approximation $u_{\mathcal{T},k}$.
- Stochastic local Kato's inequality : (with μ limit of (u_ϵ) or $(u_{\mathcal{T},k})$...)

$$\mathbb{E}\Big[\int_0^T \int_{B(0,R+C_{\vec{f}}T)} \int_{(0,1)^2} |\mu(x,t,\alpha) - \tilde{\mu}(x,t,\beta)| d\alpha d\beta dx dt\Big] = 0.$$

- Uniqueness of μ and independence with respect to the variable α.
- Uniqueness of the stochastic entropy solution $u(x,t) = \int_0^1 \mu(x,t,\alpha) d\alpha$.

Main result

For any family of monotone numerical fluxes $(F_{K,L}^n)$, the approximate solution $(u_{\mathcal{T},k})$ given by the associated finite-volume scheme converges to the unique stochastic entropy solution of the problem in $L^p_{\text{loc}}\big(\Omega \times \mathbb{R}^d \times (0,T)\big)$, $\forall p < 2$.

Thank you for your attention.

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Appendix

- State of the art.
- Uniqueness argument.
- On the uniform integrability
- Compatibility of the the stochastic integral

State of the art

- **Theoretical studies**
  - **« Strong » entropy approach** : Feng & Nualart ('08), Chen, Ding & Karlsen ('12), Biswas & Majee ('14).
  - **Kinetic approach** : Debussche & Vovelle ('10), Hofmanová ('14), Kobayasi & Noboriguchi ('16).
  - **Entropy approach** : Bauzet, Vallet & Wittbold ('12, '14).
- **Numerical analysis**
  - **Time-splitting operator method** ($d = 1$ and $d \geqslant 1$) : Holden & Risebro ('91), Bauzet ('13), Karlsen & Storrøsten(preprint).
  - **Semi-discrete Finite Volume Scheme** ($d = 1$) : Kröker & Rohde ('12), Koley, Majee & Vallet ('16).
  - **Flux-splitting FVS, monotone FVS** ($d \geqslant 1$), entropy approach : Bauzet, Charrier & Gallouët ('16, '17), Funaki, Gao & Hilhorst ('18).
  - **Monotone FVS** ($d \geqslant 1$), kinetic approach : Dotti & Vovelle ('18, '20).

Uniqueness argument

- With a particular choice of φ one gets **Kato inequality :**

$$\mathbb{E}\Big[\int_0^T \int_{B(0,R+C_{\vec{f}}T)} \int_{(0,1)^2} |\mu_1(x,t,\alpha) - \mu_2(x,t,\beta)| d\alpha d\beta dx dt\Big] = 0.$$

$\Rightarrow$ For a.a (x,t,α,β), $\mu_1(x,t,\alpha) = \mu_2(x,t,\beta)$.

- « Stochastic version » of doubling variables with $(u_{\mathcal{T},k}) \to \mu_1$ and μ_1 :

$\Rightarrow$ For a.a (x,t,α,β), $\mu_1(x,t,\alpha) = \mu_1(x,t,\beta)$.

- Set $u(x,t) = \int_0^1 \mu_1(x,t,\alpha) d\alpha$

$\Rightarrow$ For a.a (x,t,β), $u(x,t) = \int_0^1 \mu_1(x,t,\beta) d\alpha = \mu_1(x,t,\beta)$.

Conclusion : μ_1 is independent of α and a unique entropy solution u exists.

On the uniform integrability

Definition

A sequence $(\psi_n)_{n\geq 0}$ of functions $\psi_n : \Omega \times Q \to \mathbb{R}$ is uniformly integrable if

- $(\psi_n)_{n\geq 0}$ is bounded in $L^1(\Omega \times Q)$.
- $(\psi_n)_{n\geq 0}$ is equi-integrable : $\forall \varepsilon > 0,\ \exists \delta > 0 : \forall A \subset \Omega \times Q\ :$

$$(\mathcal{L}^{d+1} \otimes P)(A) \leqslant \delta \Rightarrow \forall n \in \mathbb{N}, \int_A |\psi_n(\omega, x, t)| dxdtdP \leqslant \varepsilon.$$

- $\forall \varepsilon > 0,\ \exists K_\varepsilon \subset \Omega \times Q :\ (\mathcal{L}^{d+1} \otimes P)(K_\epsilon) < \infty$ and

$$\forall n \in \mathbb{N}, \int_{K_\varepsilon^c} |\psi_n(\omega, x, t)| dxdtdP \leqslant \varepsilon.$$

Criteria

If $(\psi_n)_{n\geq 0}$ is bounded in $L^2(\Omega \times Q)$ then for any $\varphi \in L^2(\Omega \times Q)$, the product $(\psi_n \varphi)_{n\geq 0}$ is uniformly integrable.

Compatibility of the stochastic integral

Set Ψ a Carathéodory function such that $\Psi(., u_{\mathcal{T},k})$ is bounded in $L^2(\Omega \times Q)$. Then for any $\varphi \in L^2(\Omega \times Q)$

$$\mathbb{E}\Big[\int_Q \Psi(., u_{\mathcal{T},k}(x,t))\varphi dxdt\Big] \xrightarrow[h\to 0]{} \mathbb{E}\Big[\int_Q \int_0^1 \Psi(., \mathbf{u}(x,t,\alpha)) d\alpha \varphi dtdx\Big],$$

and $\Psi(., u_{\mathcal{T},k}) \rightharpoonup \int_0^1 \Psi(., \mathbf{u}(., \alpha))\, \mathrm{d}\alpha$ in $L^2(\Omega \times Q)$.

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$$\begin{aligned}\mathbf{I}:\mathcal{N}_w^2(0,T;L^2(\mathbb{R}^d)) &\to L^2(\Omega\times\mathbb{R}^d)\\ \varphi &\mapsto \int_0^T\varphi(s)\,\mathrm{d}W(s)\end{aligned}\qquad \boxed{\text{Linear and continuous}}$$

$$\Rightarrow\ \forall\phi\in L^2(\Omega\times\mathbb{R}^d),\ \mathbb{E}\Big[\int_{\mathbb{R}^d}\phi(\omega,x)\int_0^T\Psi(.,u_{\mathcal{T},k})\,\mathrm{d}W(t)dx\Big]$$

$$\xrightarrow[h\to 0]{} E\Big[\int_{\mathbb{R}^d}\phi(\omega,x)\int_0^T\int_0^1\Psi(.,\mathbf{u}(.,\alpha))\,\mathrm{d}\alpha\,\mathrm{d}W(t)dx\Big].$$