Musielak-Orlicz spaces - tool for nonlinear PDEs

Aneta Wróblewska-Kamińska

Institute of Mathematics, Polish Academy of Sciences

August 2023, Marseille

- Motivations

  Fluids which properties, as viscosity, change (significantly) under various stimuli: shear rate, electric field, magnetic field

- Motivations

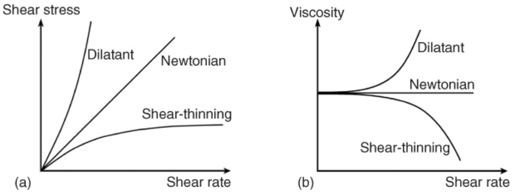
  Fluids which properties, as viscosity, change (significantly) under various stimuli: shear rate, electric field, magnetic field

- Mathematical goal

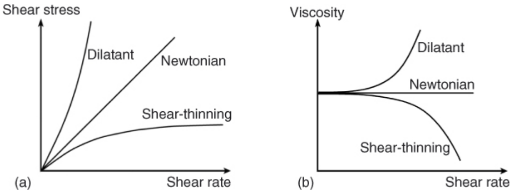
  The existence and properties of solutions to nonlinear PDE arising from dynamics of fluids of nonstandard rheology and abstract problems, where the nonlinear term of the highest order is:

  - monotone
  - growth and coercivity conditions are given with help of general convex functions

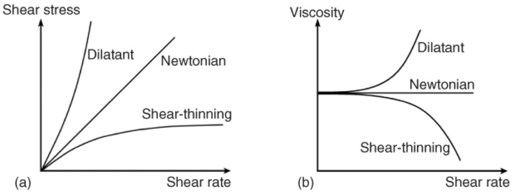
Non-Newtonian fluids

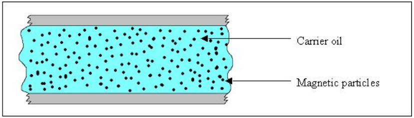


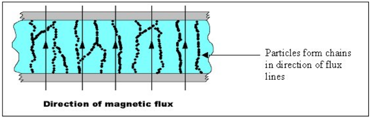
Non-Newtonian fluids



Non-Newtonian fluids







Nonlinear PDEs

$$
\begin{aligned}
\partial_t \varrho + \operatorname{div}_x(\varrho \boldsymbol{u}) &= 0 \quad \text{in} \quad I \times \Omega \\
\partial_t(\varrho \boldsymbol{u}) + \operatorname{div}_x(\varrho \boldsymbol{u} \otimes \boldsymbol{u}) - \operatorname{div}_x \mathbf{S}(x, \varrho, \mathbf{D}\boldsymbol{u}) + \nabla_x p &= \varrho \boldsymbol{f} \quad \text{in} \quad I \times \Omega \\
\operatorname{div}_x \boldsymbol{u} &= 0 \quad \text{in} \quad I \times \Omega, \\
\boldsymbol{u}(0, x) &= \boldsymbol{u}_0 \quad \text{in} \quad \Omega, \\
\varrho(0, x) &= \varrho_0 \quad \text{in} \quad \Omega, \\
\boldsymbol{u}(t, x) &= 0 \quad \text{on} \quad I \times \partial\Omega
\end{aligned}
$$

where $\Omega \subset \mathbb{R}^d$ is an open and bounded domain, $Q := I \times \Omega$,

Nonlinear PDEs

$$
\begin{aligned}
\partial_t \varrho + \operatorname{div}_x(\varrho \boldsymbol{u}) &= 0 \quad \text{in} \quad I \times \Omega \\
\partial_t(\varrho \boldsymbol{u}) + \operatorname{div}_x(\varrho \boldsymbol{u} \otimes \boldsymbol{u}) - \operatorname{div}_x \mathbf{S}(x, \varrho, \mathbf{D}\boldsymbol{u}) + \nabla_x p &= \varrho \boldsymbol{f} \quad \text{in} \quad I \times \Omega \\
\operatorname{div}_x \boldsymbol{u} &= 0 \quad \text{in} \quad I \times \Omega, \\
\boldsymbol{u}(0, x) &= \boldsymbol{u}_0 \quad \text{in} \quad \Omega, \\
\varrho(0, x) &= \varrho_0 \quad \text{in} \quad \Omega, \\
\boldsymbol{u}(t, x) &= 0 \quad \text{on} \quad I \times \partial\Omega
\end{aligned}
$$

where $\Omega \subset \mathbb{R}^d$ is an open and bounded domain, $Q := I \times \Omega$,
$\varrho : Q \to \mathbb{R}$ - denotes density
$\boldsymbol{u} : Q \to \mathbb{R}^d$ - velocity field

Nonlinear PDEs

$$\begin{aligned}
\partial_t \varrho + \mathrm{div}_x(\varrho \boldsymbol{u}) &= 0 \quad \text{in} \quad I \times \Omega \\
\partial_t(\varrho \boldsymbol{u}) + \mathrm{div}_x(\varrho \boldsymbol{u} \otimes \boldsymbol{u}) - \mathrm{div}_x \mathbf{S}(x, \varrho, \mathbf{D}\boldsymbol{u}) + \nabla_x p &= \varrho \boldsymbol{f} \quad \text{in} \quad I \times \Omega \\
\mathrm{div}_x \boldsymbol{u} &= 0 \quad \text{in} \quad I \times \Omega, \\
\boldsymbol{u}(0, x) &= \boldsymbol{u}_0 \quad \text{in} \quad \Omega, \\
\varrho(0, x) &= \varrho_0 \quad \text{in} \quad \Omega, \\
\boldsymbol{u}(t, x) &= 0 \quad \text{on} \quad I \times \partial\Omega
\end{aligned}$$

where $\Omega \subset \mathbb{R}^d$ is an open and bounded domain, $Q := I \times \Omega$,
$\varrho : Q \to \mathbb{R}$ - denotes density
$\boldsymbol{u} : Q \to \mathbb{R}^d$ - velocity field
$p : Q \to \mathbb{R}$ - pressure function
$\boldsymbol{f} : Q \to \mathbb{R}^d$ - given outer forces

Nonlinear PDEs

$$
\begin{aligned}
\partial_t \varrho + \operatorname{div}_x(\varrho \boldsymbol{u}) &= 0 \quad \text{in} \quad I \times \Omega \\
\partial_t(\varrho \boldsymbol{u}) + \operatorname{div}_x(\varrho \boldsymbol{u} \otimes \boldsymbol{u}) - \operatorname{div}_x \mathbf{S}(x, \varrho, \mathbf{D}\boldsymbol{u}) + \nabla_x p &= \varrho \boldsymbol{f} \quad \text{in} \quad I \times \Omega \\
\operatorname{div}_x \boldsymbol{u} &= 0 \quad \text{in} \quad I \times \Omega, \\
\boldsymbol{u}(0, x) &= \boldsymbol{u}_0 \quad \text{in} \quad \Omega, \\
\varrho(0, x) &= \varrho_0 \quad \text{in} \quad \Omega, \\
\boldsymbol{u}(t, x) &= 0 \quad \text{on} \quad I \times \partial\Omega
\end{aligned}
$$

where $\Omega \subset \mathbb{R}^d$ is an open and bounded domain, $Q := I \times \Omega$,
$\varrho : Q \to \mathbb{R}$ - denotes density
$\boldsymbol{u} : Q \to \mathbb{R}^d$ - velocity field
$p : Q \to \mathbb{R}$ - pressure function
$\boldsymbol{f} : Q \to \mathbb{R}^d$ - given outer forces
$\mathbf{S} : \Omega \times \mathbb{R}_+ \times \mathbb{R}^{d\times d} \to \mathbb{R}^{d\times d}$ - stress tensor
$\mathbf{D}\boldsymbol{u} = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla^T \boldsymbol{u})$

Non-Newtonian fluids – classical growth condition – power-law

$$\partial_t \boldsymbol{u} - \operatorname{div}_x \mathbf{S}(x, \mathbf{D}\boldsymbol{u}) + \dots = \boldsymbol{f}$$

- standard growth and coercivity conditions
  - $$|\mathbf{S}(x, \mathbf{D}\boldsymbol{u})| \leqslant c_1 |\mathbf{D}\boldsymbol{u}|^{p-1}$$
  - $$\mathbf{S}(x, \mathbf{D}\boldsymbol{u}) : \mathbf{D}\boldsymbol{u} \geqslant c_2 |\mathbf{D}\boldsymbol{u}|^{p}$$

Non-Newtonian fluids – classical growth condition – power-law

$$\partial_t \boldsymbol{u} - \operatorname{div}_x \mathbf{S}(x, \mathbf{D}\boldsymbol{u}) + \cdots = \boldsymbol{f}$$

- standard growth and coercivity conditions
  - $$|\mathbf{S}(x, \mathbf{D}\boldsymbol{u})| \leqslant c_1 |\mathbf{D}\boldsymbol{u}|^{p-1}$$
  - $$\mathbf{S}(x, \mathbf{D}\boldsymbol{u}) : \mathbf{D}\boldsymbol{u} \geqslant c_2 |\mathbf{D}\boldsymbol{u}|^{p}$$
- $p = 2$ newtonian fluids
- $p > 2$ shear thickening, $p < 2$ shear thinning

Non-Newtonian fluids – classical growth condition – power-law

$$\partial_t \boldsymbol{u} - \operatorname{div}_x \mathbf{S}(x, \mathbf{D}\boldsymbol{u}) + \cdots = \boldsymbol{f}$$

- standard growth and coercivity conditions
  - $$|\mathbf{S}(x, \mathbf{D}\boldsymbol{u})| \leqslant c_1 |\mathbf{D}\boldsymbol{u}|^{p-1}$$
  - $$\mathbf{S}(x, \mathbf{D}\boldsymbol{u}) : \mathbf{D}\boldsymbol{u} \geqslant c_2 |\mathbf{D}\boldsymbol{u}|^{p}$$
- $p = 2$ newtonian fluids
- $p > 2$ shear thickening, $p < 2$ shear thinning
- Lebesgue and Sobolev spaces

Non-Newtonian fluids – classical growth condition – power-law

$$\partial_t \boldsymbol{u} - \mathrm{div}_x \mathbf{S}(x, \mathbf{D}\boldsymbol{u}) + \cdots = \boldsymbol{f}$$

- standard growth and coercivity conditions
  - $$|\mathbf{S}(x, \mathbf{D}\boldsymbol{u})| \leqslant c_1 |\mathbf{D}\boldsymbol{u}|^{p-1}$$
  - $$\mathbf{S}(x, \mathbf{D}\boldsymbol{u}) : \mathbf{D}\boldsymbol{u} \geqslant c_2 |\mathbf{D}\boldsymbol{u}|^{p}$$
- $p = 2$ newtonian fluids
- $p > 2$ shear thickening, $p < 2$ shear thinning
- Lebesgue and Sobolev spaces
- J. Nečas, J. Málek, M. Bulíček, J. Frehse, L. Diening, J. Wolf, M. Růžička, M. Fuchs, P. Kaplický, Beiro de Veiga, and ...

Generalizations

Let us take instead of $M(\xi) = |\xi|^p$ a more general function: convex, continuous and of super-linear growth

Generalizations

Let us take instead of $M(\xi) = |\xi|^p$ a more general function: convex, continuous and of super-linear growth

- rapidly or slowly growing

$$\exp(|\xi|) - 1, \quad |\xi| \ln(|\xi| + 1)$$

Generalizations

Let us take instead of $M(\xi) = |\xi|^p$ a more general function: convex, continuous and of super-linear growth

- rapidly or slowly growing

$$\exp(|\xi|) - 1, \quad |\xi| \ln(|\xi| + 1)$$

- anizotropic, i.e. $M : \mathbb{R}^d \to \mathbb{R}_+$

$$\frac{1}{p_1}|\xi_1|^{p_1} + \frac{1}{p_2}|\xi_2|^{p_2} + \cdots + \frac{1}{p_d}|\xi_d|^{p_d}$$

Generalizations

Let us take instead of $M(\xi) = |\xi|^p$ a more general function: convex, continuous and of super-linear growth

- rapidly or slowly growing

$$\exp(|\xi|) - 1, \quad |\xi| \ln(|\xi| + 1)$$

- anizotropic, i.e. $M : \mathbb{R}^d \to \mathbb{R}_+$

$$\frac{1}{p_1}|\xi_1|^{p_1} + \frac{1}{p_2}|\xi_2|^{p_2} + \cdots + \frac{1}{p_d}|\xi_d|^{p_d}$$

- non-homogenous in space, i.e. with dependence on x $M : \Omega \times \mathbb{R}^d \to \mathbb{R}_+$,

$$|\xi|^{p(x)}$$

Generalizations

Let us take instead of $M(\xi) = |\xi|^p$ a more general function: convex, continuous and of super-linear growth

- rapidly or slowly growing

$$\exp(|\xi|) - 1, \quad |\xi| \ln(|\xi| + 1)$$

- anizotropic, i.e. $M : \mathbb{R}^d \to \mathbb{R}_+$

$$\frac{1}{p_1}|\xi_1|^{p_1} + \frac{1}{p_2}|\xi_2|^{p_2} + \cdots + \frac{1}{p_d}|\xi_d|^{p_d}$$

- non-homogenous in space, i.e. with dependence on x $M : \Omega \times \mathbb{R}^d \to \mathbb{R}_+$,

$$|\xi|^{p(x)}$$

- Orlicz spaces, anisotropic Orlicz spaces, Musielak-Orlicz spaces

Generalizations

Let us take instead of $M(\xi) = |\xi|^p$ a more general function: convex, continuous and of super-linear growth

- rapidly or slowly growing

$$\exp(|\xi|) - 1, \quad |\xi| \ln(|\xi| + 1)$$

- anizotropic, i.e. $M : \mathbb{R}^d \to \mathbb{R}_+$

$$\frac{1}{p_1}|\xi_1|^{p_1} + \frac{1}{p_2}|\xi_2|^{p_2} + \cdots + \frac{1}{p_d}|\xi_d|^{p_d}$$

- non-homogenous in space, i.e. with dependence on x $M : \Omega \times \mathbb{R}^d \to \mathbb{R}_+$,

$$|\xi|^{p(x)}$$

- Orlicz spaces, anisotropic Orlicz spaces, Musielak-Orlicz spaces

General growth and coercivity condition

Our general growth and coercivity conditions:

$$\mathbf{S}(x, \mathbf{D}\boldsymbol{u}) : \mathbf{D}\boldsymbol{u} \geqslant d_1\{M(x, d_2\mathbf{D}\boldsymbol{u}) + M^*(x, d_3\mathbf{S}(x, \mathbf{D}\boldsymbol{u}))\}$$

General growth and coercivity condition

Our general growth and coercivity conditions:

$$\mathbf{S}(x, \mathbf{D}\boldsymbol{u}) : \mathbf{D}\boldsymbol{u} \geqslant d_1\{M(x, d_2\mathbf{D}\boldsymbol{u}) + M^*(x, d_3\mathbf{S}(x, \mathbf{D}\boldsymbol{u}))\}$$

Are equivalent to :

$$M(x, c_1\mathbf{D}\boldsymbol{u}) \leqslant \mathbf{S}(x, \mathbf{D}\boldsymbol{u}) : \mathbf{D}\boldsymbol{u}$$
$$c_2 M^*(x, c_3\mathbf{S}(x, \mathbf{D}\boldsymbol{u})) \leqslant M(x, c_4\mathbf{D}\boldsymbol{u})$$

N-function and conjugate

$M(x,\xi) : \Omega \times \mathbb{R}^d \to \mathbb{R}_+$ is called an $\mathcal{N}$*–function*

N-function and conjugate

$M(x, \xi) : \Omega \times \mathbb{R}^d \to \mathbb{R}_+$ is called an $\mathcal{N}$*–function*

1. M is Carathéodory, convex, nonnegative

N-function and conjugate

$M(x,\boldsymbol{\xi}) : \Omega \times \mathbb{R}^d \to \mathbb{R}_+$ is called an $\mathcal{N}$*–function*

1. M is Carathéodory, convex, nonnegative
2. $M(x,\boldsymbol{\xi}) = M(x,-\boldsymbol{\xi})$

N-function and conjugate

$M(x, \boldsymbol{\xi}) : \Omega \times \mathbb{R}^d \to \mathbb{R}_+$ is called an $\mathcal{N}$*–function*

1. M is Carathéodory, convex, nonnegative
2. $M(x, \boldsymbol{\xi}) = M(x, -\boldsymbol{\xi})$
3. $M(x, \boldsymbol{\xi}) = 0$ iff $\boldsymbol{\xi} = 0$

N-function and conjugate

$M(x,\xi) : \Omega \times \mathbb{R}^d \to \mathbb{R}_+$ is called an $\mathcal{N}$*–function*

1. M is Carathéodory, convex, nonnegative
2. $M(x,\xi) = M(x,-\xi)$
3. $M(x,\xi) = 0$ iff $\xi = 0$
4. there exist two Young functions $m_1,\ m_2 : [0,\infty) \to [0,\infty)$ such that

$$m_1(|\xi|) \leqslant M(x,\xi) \leqslant m_2(|\xi|) \quad \text{for a.a. } x \in \Omega$$

N-function and conjugate

$M(x,\xi):\Omega\times\mathbb{R}^d\to\mathbb{R}_+$ is called an $\mathcal{N}$*–function*

1. M is Carathéodory, convex, nonnegative
2. $M(x,\xi)=M(x,-\xi)$
3. $M(x,\xi)=0$ iff $\xi=0$
4. there exist two Young functions $m_1,\ m_2:[0,\infty)\to[0,\infty)$ such that

$$m_1(|\xi|)\leqslant M(x,\xi)\leqslant m_2(|\xi|)\qquad\text{for a.a. } x\in\Omega$$

$m:[0,\infty)\to[0,\infty)$ is Young function if $m(s)=0$ iff $s=0$, m is convex and super-linear at zero and infinity, i.e.

$$\lim_{s\to0^+}\frac{m(s)}{s}=0\quad\text{and}\quad\lim_{s\to\infty}\frac{m(s)}{s}=\infty$$

N-function and conjugate

$M(x,\xi):\Omega\times\mathbb{R}^d\to\mathbb{R}_+$ is called an $\mathcal{N}$*–function*

1. M is Carathéodory, convex, nonnegative
2. $M(x,\xi)=M(x,-\xi)$
3. $M(x,\xi)=0$ iff $\xi=0$
4. there exist two Young functions $m_1,\ m_2:[0,\infty)\to[0,\infty)$ such that

$$m_1(|\xi|)\leqslant M(x,\xi)\leqslant m_2(|\xi|)\qquad \text{for a.a. } x\in\Omega$$

$m:[0,\infty)\to[0,\infty)$ is Young function if $m(s)=0$ iff $s=0$, m is convex and super-linear at zero and infinity, i.e.

$$\lim_{s\to0^+}\frac{m(s)}{s}=0\quad\text{and}\quad\lim_{s\to\infty}\frac{m(s)}{s}=\infty$$

A complementary function M^* to M is defined by:

$$M^*(x,\zeta)=\sup_{\xi\in\mathbb{R}^d}\left(\xi:\zeta-M(x,\xi)\right).$$

N-function and conjugate

$M(x,\xi) : \Omega \times \mathbb{R}^d \to \mathbb{R}_+$ is called an $\mathcal{N}$*–function*

1. M is Carathéodory, convex, nonnegative
2. $M(x,\xi) = M(x,-\xi)$
3. $M(x,\xi) = 0$ iff $\xi = 0$
4. there exist two Young functions $m_1,\ m_2 : [0,\infty) \to [0,\infty)$ such that

$$m_1(|\xi|) \leqslant M(x,\xi) \leqslant m_2(|\xi|) \quad \text{for a.a. } x \in \Omega$$

$m : [0,\infty) \to [0,\infty)$ is Young function if $m(s) = 0$ iff $s = 0$, m is convex and super-linear at zero and infinity, i.e.

$$\lim_{s\to 0^+} \frac{m(s)}{s} = 0 \quad \text{and} \quad \lim_{s\to\infty} \frac{m(s)}{s} = \infty$$

A complementary function M^* to M is defined by:

$$M^*(x,\zeta) = \sup_{\xi\in\mathbb{R}^d} \left(\xi : \zeta - M(x,\xi)\right).$$

M^* is also an $\mathcal{N}$–function

Anisotropic Musielak-Orlicz spaces

Anisotropic Musielak-Orlicz spaces

Musielak-Orlicz class $\mathcal{L}_M(Q;\mathbb{R}^d)$

$$\int_Q M(x,\mathbf{f}(t,x))\,\mathrm{d}x\mathrm{d}t < \infty$$

Anisotropic Musielak-Orlicz spaces

Musielak-Orlicz class $\mathcal{L}_M(Q;\mathbb{R}^d)$

$$\int_Q M(x,\mathbf{f}(t,x))\,\mathrm{d}x\mathrm{d}t < \infty$$

Musielak-Orlicz space $L_M(Q;\mathbb{R}^d)$

$$\int_Q M(x,\lambda\mathbf{f}(t,x))\,\mathrm{d}x\mathrm{d}t \to 0 \quad \text{as}\, \lambda \to 0$$

Anisotropic Musielak-Orlicz spaces

Musielak-Orlicz class $\mathcal{L}_M(Q;\mathbb{R}^d)$

$$\int_Q M(x,\mathbf{f}(t,x))\,\mathrm{d}x\mathrm{d}t < \infty$$

Musielak-Orlicz space $L_M(Q;\mathbb{R}^d)$

$$\int_Q M(x,\lambda\mathbf{f}(t,x))\,\mathrm{d}x\mathrm{d}t \to 0 \quad \text{as}\,\lambda \to 0$$

It is a Banach space w.r.t. the Luxemburg norm

$$\|\boldsymbol{\xi}\|_M = \inf\left\{\lambda > 0 \mid \int_Q M\left(x, \frac{\boldsymbol{\xi}(t,x)}{\lambda}\right)\,\mathrm{d}x\mathrm{d}t \leqslant 1\right\}.$$

Anisotropic Musielak-Orlicz spaces

Musielak-Orlicz class $\mathcal{L}_M(Q;\mathbb{R}^d)$

$$\int_Q M(x,\mathbf{f}(t,x))\,\mathrm{d}x\mathrm{d}t < \infty$$

Musielak-Orlicz space $L_M(Q;\mathbb{R}^d)$

$$\int_Q M(x,\lambda\mathbf{f}(t,x))\,\mathrm{d}x\mathrm{d}t \to 0 \quad \text{as}\, \lambda \to 0$$

It is a Banach space w.r.t. the Luxemburg norm

$$\|\boldsymbol{\xi}\|_M = \inf\left\{\lambda > 0 \mid \int_Q M\left(x, \frac{\boldsymbol{\xi}(t,x)}{\lambda}\right)\,\mathrm{d}x\mathrm{d}t \leqslant 1\right\}.$$

$E_M(Q;\mathbb{R}^d)$ is a closure of simple functions on Q in $L_M(Q;\mathbb{R}^d)$

Anisotropic Musielak-Orlicz spaces

Musielak-Orlicz class $\mathcal{L}_M(Q;\mathbb{R}^d)$

$$\int_Q M(x,\mathbf{f}(t,x))\,\mathrm{d}x\mathrm{d}t<\infty$$

Musielak-Orlicz space $L_M(Q;\mathbb{R}^d)$

$$\int_Q M(x,\lambda\mathbf{f}(t,x))\,\mathrm{d}x\mathrm{d}t\to 0\quad \text{as}\,\lambda\to 0$$

It is a Banach space w.r.t. the Luxemburg norm

$$\|\boldsymbol{\xi}\|_M=\inf\left\{\lambda>0\mid\int_Q M\left(x,\frac{\boldsymbol{\xi}(t,x)}{\lambda}\right)\,\mathrm{d}x\mathrm{d}t\leqslant 1\right\}.$$

$E_M(Q;\mathbb{R}^d)$ is a closure of simple functions on Q in $L_M(Q;\mathbb{R}^d)$

- $E_M\subseteq\mathcal{L}_M\subseteq L_M$

Anisotropic Musielak-Orlicz spaces

Musielak-Orlicz class $\mathcal{L}_M(Q;\mathbb{R}^d)$

$$\int_Q M(x,\mathbf{f}(t,x))\,\mathrm{d}x\mathrm{d}t < \infty$$

Musielak-Orlicz space $L_M(Q;\mathbb{R}^d)$

$$\int_Q M(x,\lambda\mathbf{f}(t,x))\,\mathrm{d}x\mathrm{d}t \to 0 \quad \text{as}\ \lambda \to 0$$

It is a Banach space w.r.t. the Luxemburg norm

$$\|\boldsymbol{\xi}\|_M = \inf\left\{\lambda > 0 \mid \int_Q M\left(x, \frac{\boldsymbol{\xi}(t,x)}{\lambda}\right)\,\mathrm{d}x\mathrm{d}t \leqslant 1\right\}.$$

$E_M(Q;\mathbb{R}^d)$ is a closure of simple functions on Q in $L_M(Q;\mathbb{R}^d)$

- $E_M \subseteq \mathcal{L}_M \subseteq L_M$
- $(E_M)^* = L_{M^*}$

We say that the $\mathcal{N}$–function M satisfies *Δ_2–condition*, if

$$M(x, 2\boldsymbol{\xi}) \leqslant C_M M(x, \boldsymbol{\xi}) + g_M(x)$$

for all $\boldsymbol{\xi} \in \mathbb{R}^d$ and some integrable nonnegative function g_M

Anisotropic Musielak-Orlicz spaces

We say that the $\mathcal{N}$–function M satisfies *Δ_2–condition*, if

$$M(x, 2\boldsymbol{\xi}) \leqslant C_M M(x, \boldsymbol{\xi}) + g_M(x)$$

for all $\boldsymbol{\xi} \in \mathbb{R}^d$ and some integrable nonnegative function g_M

If M does NOT satisfy Δ_2–condition then:

Anisotropic Musielak-Orlicz spaces

We say that the $\mathcal{N}$–function M satisfies *Δ_2–condition*, if

$$M(x, 2\boldsymbol{\xi}) \leqslant C_M M(x, \boldsymbol{\xi}) + g_M(x)$$

for all $\boldsymbol{\xi} \in \mathbb{R}^d$ and some integrable nonnegative function g_M

If M does NOT satisfy Δ_2–condition then:

- L_M is not separable

Anisotropic Musielak-Orlicz spaces

We say that the $\mathcal{N}$–function M satisfies *Δ_2–condition*, if

$$M(x, 2\xi) \leqslant C_M M(x, \xi) + g_M(x)$$

for all $\xi \in \mathbb{R}^d$ and some integrable nonnegative function g_M

If M does NOT satisfy Δ_2–condition then:

- L_M is not separable
- C^∞ is not dense in L_M

Anisotropic Musielak-Orlicz spaces

We say that the $\mathcal{N}$–function M satisfies *Δ_2–condition*, if

$$M(x, 2\boldsymbol{\xi}) \leqslant C_M M(x, \boldsymbol{\xi}) + g_M(x)$$

for all $\boldsymbol{\xi} \in \mathbb{R}^d$ and some integrable nonnegative function g_M

If M does NOT satisfy Δ_2–condition then:

- L_M is not separable
- C^∞ is not dense in L_M
- L_M is not reflexive

Anisotropic Musielak-Orlicz spaces

We say that the $\mathcal{N}$–function M satisfies *Δ_2–condition*, if

$$M(x, 2\boldsymbol{\xi}) \leqslant C_M M(x, \boldsymbol{\xi}) + g_M(x)$$

for all $\boldsymbol{\xi} \in \mathbb{R}^d$ and some integrable nonnegative function g_M

If M does NOT satisfy Δ_2–condition then:

- L_M is not separable
- C^∞ is not dense in L_M
- L_M is not reflexive
- $E_M \subsetneq \mathcal{L}_M \subsetneq L_M$

Anisotropic Musielak-Orlicz spaces

We say that the $\mathcal{N}$–function M satisfies *Δ_2–condition*, if

$$M(x, 2\boldsymbol{\xi}) \leqslant C_M M(x, \boldsymbol{\xi}) + g_M(x)$$

for all $\boldsymbol{\xi} \in \mathbb{R}^d$ and some integrable nonnegative function g_M

If M does NOT satisfy Δ_2–condition then:

- L_M is not separable
- C^∞ is not dense in L_M
- L_M is not reflexive
- $E_M \subsetneq \mathcal{L}_M \subsetneq L_M$

Some other problems:

- $L_M(Q) \neq L_M(0, T; L_M(\Omega))$

Anisotropic Musielak-Orlicz spaces

We say that the $\mathcal{N}$–function M satisfies *Δ_2–condition*, if

$$M(x, 2\xi) \leqslant C_M M(x, \xi) + g_M(x)$$

for all $\xi \in \mathbb{R}^d$ and some integrable nonnegative function g_M

If M does NOT satisfy Δ_2–condition then:

- L_M is not separable
- C^∞ is not dense in L_M
- L_M is not reflexive
- $E_M \subsetneq \mathcal{L}_M \subsetneq L_M$

Some other problems:

- $L_M(Q) \neq L_M(0, T; L_M(\Omega))$
- singular operators may be not continuous from L_M to L_M

We say that the $\mathcal{N}$–function M satisfies *Δ_2–condition*, if

$$M(x, 2\xi) \leqslant C_M M(x, \xi) + g_M(x)$$

for all $\xi \in \mathbb{R}^d$ and some integrable nonnegative function g_M

If M does NOT satisfy Δ_2–condition then:

- L_M is not separable
- C^∞ is not dense in L_M
- L_M is not reflexive
- $E_M \subsetneq \mathcal{L}_M \subsetneq L_M$

Some other problems:

- $L_M(Q) \neq L_M(0, T; L_M(\Omega))$
- singular operators may be not continuous from L_M to L_M
- smooth functions may not be dense in a modular topology if M depends on x

Assumptions on the nonlinear term

We assume that the term $\mathbf{S}$ satisfies:

Assumptions on the nonlinear term

We assume that the term $\mathbf{S}$ satisfies:

(S1) $\mathbf{S}(x, \mathbf{D}\boldsymbol{u})$ is a Carathéodory function

Assumptions on the nonlinear term

We assume that the term $\mathbf{S}$ satisfies:

(S1) $\mathbf{S}(x, \mathbf{D}\boldsymbol{u})$ is a Carathéodory function

(S2) Coercivity/growth condtion

$$\mathbf{S}(x, \boldsymbol{\xi}) : \boldsymbol{\xi} \geqslant c\{M(x, \boldsymbol{\xi}) + M^*(x, \mathbf{S}(x, \boldsymbol{\xi}))\}$$

Assumptions on the nonlinear term

We assume that the term $\mathbf{S}$ satisfies:

(S1) $\mathbf{S}(x, \mathbf{D}\boldsymbol{u})$ is a Carathéodory function

(S2) Coercivity/growth condtion

$$\mathbf{S}(x,\boldsymbol{\xi}) : \boldsymbol{\xi} \geqslant c\{M(x,\boldsymbol{\xi}) + M^*(x, \mathbf{S}(x,\boldsymbol{\xi}))\}$$

(S3) $\mathbf{S}$ is monotonicity - for all $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ and a.a. $x \in \Omega$

$$(\mathbf{S}(x,\boldsymbol{\xi}) - \mathbf{S}(x,\boldsymbol{\eta})) : (\boldsymbol{\xi} - \boldsymbol{\eta}) \geqslant 0$$

Assumptions on the nonlinear term

We assume that the term $\mathbf{S}$ satisfies:

(S1) $\mathbf{S}(x, \mathbf{D}\boldsymbol{u})$ is a Carathéodory function

(S2) Coercivity/growth condtion

$$\mathbf{S}(x,\boldsymbol{\xi}) : \boldsymbol{\xi} \geqslant c\{M(x,\boldsymbol{\xi}) + M^*(x,\mathbf{S}(x,\boldsymbol{\xi}))\}$$

(S3) $\mathbf{S}$ is monotonicity - for all $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ and a.a. $x \in \Omega$

$$(\mathbf{S}(x,\boldsymbol{\xi}) - \mathbf{S}(x,\boldsymbol{\eta})) : (\boldsymbol{\xi} - \boldsymbol{\eta}) \geqslant 0$$

(S4) Additional assumptions on an $\mathcal{N}-$function M depends on the particular problem

Results - non-Newtonian fluids: existence of weak solutions.

Shear thickening fluids

Shear thickening fluids

$$
\begin{aligned}
\partial_t \varrho + \operatorname{div}_x(\varrho \boldsymbol{u}) &= 0 \quad \text{in} \quad Q \\
\partial_t(\varrho \boldsymbol{u}) + \operatorname{div}_x(\varrho \boldsymbol{u} \otimes \boldsymbol{u}) - \operatorname{div}_x \mathbf{S}(x, \mathbf{D}\boldsymbol{u}) + \nabla_x p &= \varrho \boldsymbol{f} \quad \text{in} \quad Q \\
\operatorname{div}_x \boldsymbol{u} &= 0 \quad \text{in} \quad Q, \\
\boldsymbol{u}(t, x) &= 0 \quad \text{on} \quad I \times \partial\Omega
\end{aligned}
$$

Results - non-Newtonian fluids: existence of weak solutions.

Shear thickening fluids

$$
\begin{aligned}
\partial_t \varrho + \mathrm{div}_x(\varrho \boldsymbol{u}) &= 0 \quad \text{in} \quad Q \\
\partial_t(\varrho \boldsymbol{u}) + \mathrm{div}_x(\varrho \boldsymbol{u} \otimes \boldsymbol{u}) - \mathrm{div}_x \mathbf{S}(x, \mathbf{D}\boldsymbol{u}) + \nabla_x p &= \varrho \boldsymbol{f} \quad \text{in} \quad Q \\
\mathrm{div}_x \boldsymbol{u} &= 0 \quad \text{in} \quad Q, \\
\boldsymbol{u}(t,x) &= 0 \quad \text{on} \quad I \times \partial\Omega
\end{aligned}
$$

1. the flow of a non-homogeneous non-Newtonian incompressible fluid with a nonstandard rheology: $\mathbf{S} = \mathbf{S}(x, \varrho, \mathbf{D}\boldsymbol{u})$ and

$$
M(x, \boldsymbol{\xi}) \geqslant |\boldsymbol{\xi}|^p, \quad p \geqslant \frac{11}{5}
$$

Results - non-Newtonian fluids: existence of weak solutions.

Shear thickening fluids

$$
\begin{aligned}
\partial_t \varrho + \operatorname{div}_x(\varrho \boldsymbol{u}) &= 0 \quad \text{in} \quad Q \\
\partial_t(\varrho \boldsymbol{u}) + \operatorname{div}_x(\varrho \boldsymbol{u} \otimes \boldsymbol{u}) - \operatorname{div}_x \mathbf{S}(x, \mathbf{D}\boldsymbol{u}) + \nabla_x p &= \varrho \boldsymbol{f} \quad \text{in} \quad Q \\
\operatorname{div}_x \boldsymbol{u} &= 0 \quad \text{in} \quad Q, \\
\boldsymbol{u}(t,x) &= 0 \quad \text{on} \quad I \times \partial\Omega
\end{aligned}
$$

1. the flow of a non-homogeneous non-Newtonian incompressible fluid with a nonstandard rheology: $\mathbf{S} = \mathbf{S}(x, \varrho, \mathbf{D}\boldsymbol{u})$ and

$$
M(x, \boldsymbol{\xi}) \geqslant |\boldsymbol{\xi}|^p, \quad p \geqslant \frac{11}{5}
$$

$\boldsymbol{u} \in L^p(0, T; W^{1,p}_{0,\mathrm{div}})$, $\mathbf{D}\boldsymbol{u} \in L_M(Q)$, $\mathbf{S} \in L_{M^*}(Q)$

A.WK. DCDS, 2013

Results - non-Newtonian fluids: existence of weak solutions.

Shear thickening fluids

$$\begin{aligned} \partial_t \varrho + \operatorname{div}_x(\varrho \boldsymbol{u}) &= 0 \quad \text{in } Q \\ \partial_t(\varrho \boldsymbol{u}) + \operatorname{div}_x(\varrho \boldsymbol{u} \otimes \boldsymbol{u}) - \operatorname{div}_x \mathbf{S} + \nabla_x p &= \varrho \boldsymbol{f} \\ \partial_t(\varrho\theta) + \operatorname{div}_x(\varrho\theta \boldsymbol{u}) - \operatorname{div}_x \boldsymbol{q} &= \mathbf{S} : \mathbf{D}\boldsymbol{u} \\ \operatorname{div}_x \boldsymbol{u} &= 0 \\ \boldsymbol{q} \cdot \boldsymbol{n} = 0, \quad \boldsymbol{u}(t,x) &= 0 \quad \text{on } I \times \partial\Omega \end{aligned}$$

$\theta : Q \to \mathbb{R}$ - temperature,
$\mathbf{S} = \mathbf{S}(x, \varrho, \theta, \mathbf{D}\boldsymbol{u})$
$\boldsymbol{q} = \boldsymbol{q}(\varrho, \theta, \nabla_x \theta) : \mathbb{R}^2_+ \times \mathbb{R}^3 \to \mathbb{R}^3$ - heat flux

3. the flow of a non-homogeneous non-Newtonian heat-conducting incompressible fluids - $M(x, \boldsymbol{\xi}) \geqslant |\boldsymbol{\xi}|^p$, $p \geqslant \frac{11}{5}$, $\boldsymbol{q}$ has a linear growth w.r.t. $\nabla\theta$.

A.WK. B. Matejczyk. Nonlinearity 2018.

Main steps in the proof. The beginning is quite standard (non-Newtonian fluids) ...

Main steps in the proof. The beginning is quite standard (non-Newtonian fluids) ...

- Constructing a family of approximate solutions

Main steps in the proof. The beginning is quite standard (non-Newtonian fluids) ...

- Constructing a family of approximate solutions
- Uniform bounds

Main steps in the proof. The beginning is quite standard (non-Newtonian fluids) ...

- Constructing a family of approximate solutions
- Uniform bounds
- Theory of renormalized solutions for a transport equation (Lions, DiPerna)

Main steps in the proof. The beginning is quite standard (non-Newtonian fluids) ...

- Constructing a family of approximate solutions
- Uniform bounds
- Theory of renormalized solutions for a transport equation (Lions, DiPerna)
- Compensated compactness methods, Curl-Div Lemma (Tartar, Feireisl)

Main steps in the proof. The beginning is quite standard (non-Newtonian fluids) ...

- Constructing a family of approximate solutions
- Uniform bounds
- Theory of renormalized solutions for a transport equation (Lions, DiPerna)
- Compensated compactness methods, Curl-Div Lemma (Tartar, Feireisl)
- For the convective term: as $p \geqslant \frac{11}{5}$, $\boldsymbol{u}^n$ converges strongly in L^2

Main steps in the proof. The beginning is quite standard (non-Newtonian fluids) ...

- Constructing a family of approximate solutions
- Uniform bounds
- Theory of renormalized solutions for a transport equation (Lions, DiPerna)
- Compensated compactness methods, Curl-Div Lemma (Tartar, Feireisl)
- For the convective term: as $p \geqslant \frac{11}{5}$, $\boldsymbol{u}^n$ converges strongly in L^2
- ...

Nonlinear term

The main difficulty we have to face in the proof of sequential stability of approximate solutions is

- convergence in the nonlinear term

$$\partial_t \boldsymbol{u}^n - \operatorname{div}_x \mathbf{S}(x, \mathbf{D}\boldsymbol{u}^n) + \cdots = \boldsymbol{f}$$

Nonlinear term

The main difficulty we have to face in the proof of sequential stability of approximate solutions is

- convergence in the nonlinear term

$$\partial_t \boldsymbol{u}^n - \mathrm{div}_x\, \mathbf{S}(x, \mathbf{D}\boldsymbol{u}^n) + \dots = \boldsymbol{f}$$

- From energy estimates we obtain

$$\mathbf{D}\boldsymbol{u}^n \overset{*}{\rightharpoonup} \mathbf{D}\boldsymbol{u} \text{ in } L_M \quad \text{and} \quad \mathbf{S}(x, \mathbf{D}\boldsymbol{u}^n) \overset{*}{\rightharpoonup} \boldsymbol{\chi} \text{ in } L_{M^*}$$

Nonlinear term

The main difficulty we have to face in the proof of sequential stability of approximate solutions is

- convergence in the nonlinear term

$$\partial_t \boldsymbol{u}^n - \mathrm{div}_x \, \mathbf{S}(x, \mathbf{D}\boldsymbol{u}^n) + \cdots = \boldsymbol{f}$$

- From energy estimates we obtain

$$\mathbf{D}\boldsymbol{u}^n \overset{*}{\rightharpoonup} \mathbf{D}\boldsymbol{u} \text{ in } L_M \quad \text{and} \quad \mathbf{S}(x, \mathbf{D}\boldsymbol{u}^n) \overset{*}{\rightharpoonup} \boldsymbol{\chi} \text{ in } L_{M^*}$$

- The main question:

$$\text{Is } \boldsymbol{\chi} = \mathbf{S}(x, \mathbf{D}\boldsymbol{u})?$$

Monotonicity method to show that $\boldsymbol{\chi} = \mathbf{S}(x, \mathbf{D}\boldsymbol{u})$

Monotonicity method to show that $\chi = \mathbf{S}(x, \mathbf{D}\boldsymbol{u})$

- $$(\mathbf{S}(x, \mathbf{w}) - \mathbf{S}(x, \mathbf{D}\boldsymbol{u}^n)) : (\mathbf{w} - \mathbf{D}\boldsymbol{u}^n) \geqslant 0$$

Monotonicity method to show that $\chi = \mathbf{S}(x, \mathbf{D}\boldsymbol{u})$

- $$(\mathbf{S}(x,\mathbf{w}) - \mathbf{S}(x,\mathbf{D}\boldsymbol{u}^n)) : (\mathbf{w} - \mathbf{D}\boldsymbol{u}^n) \geqslant 0$$

- $$\mathbf{D}\boldsymbol{u}^n \overset{*}{\rightharpoonup} \mathbf{D}\boldsymbol{u} \text{ in } L_M \quad \text{and} \quad \mathbf{S}(x,\mathbf{D}\boldsymbol{u}^n) \overset{*}{\rightharpoonup} \chi \text{ in } L_{M^*}$$

Monotonicity method to show that $\chi = \mathbf{S}(x, \mathbf{D}\boldsymbol{u})$

- $$(\mathbf{S}(x, \mathbf{w}) - \mathbf{S}(x, \mathbf{D}\boldsymbol{u}^n)) : (\mathbf{w} - \mathbf{D}\boldsymbol{u}^n) \geqslant 0$$

- $$\mathbf{D}\boldsymbol{u}^n \overset{*}{\rightharpoonup} \mathbf{D}\boldsymbol{u} \text{ in } L_M \quad \text{and} \quad \mathbf{S}(x, \mathbf{D}\boldsymbol{u}^n) \overset{*}{\rightharpoonup} \chi \text{ in } L_{M^*}$$

- Since $\mathbf{D}\boldsymbol{u}^n \in L_M$ and $L_M \neq E_M$, $\mathbf{w} = \mathbf{D}\boldsymbol{u} + h\boldsymbol{v} \in L_M$ is not a proper test function

Monotonicity method to show that $\chi = \mathbf{S}(x, \mathbf{D}\boldsymbol{u})$

- $$(\mathbf{S}(x, \mathbf{w}) - \mathbf{S}(x, \mathbf{D}\boldsymbol{u}^n)) : (\mathbf{w} - \mathbf{D}\boldsymbol{u}^n) \geqslant 0$$
- $$\mathbf{D}\boldsymbol{u}^n \overset{*}{\rightharpoonup} \mathbf{D}\boldsymbol{u} \text{ in } L_M \quad \text{and} \quad \mathbf{S}(x, \mathbf{D}\boldsymbol{u}^n) \overset{*}{\rightharpoonup} \chi \text{ in } L_{M^*}$$
- Since $\mathbf{D}\boldsymbol{u}^n \in L_M$ and $L_M \neq E_M$, $\mathbf{w} = \mathbf{D}\boldsymbol{u} + h\boldsymbol{v} \in L_M$ is not a proper test function
- Monotonicity methods and Browder-Minty trick adapted to non-reflexive Orlicz spaces:

$$\mathbf{w} = \mathbf{D}\boldsymbol{u}\mathbb{1}_{\{|\mathbf{D}\boldsymbol{u}|<j\}} + h\boldsymbol{v}\mathbb{1}_{\{|\mathbf{D}\boldsymbol{u}|<i\}}, \;\; \boldsymbol{v} \in L^\infty$$

Monotonicity method to show that $\chi = \mathbf{S}(x, \mathbf{D}\boldsymbol{u})$

- $$(\mathbf{S}(x,\mathbf{w}) - \mathbf{S}(x,\mathbf{D}\boldsymbol{u}^n)) : (\mathbf{w} - \mathbf{D}\boldsymbol{u}^n) \geqslant 0$$
- $$\mathbf{D}\boldsymbol{u}^n \overset{*}{\rightharpoonup} \mathbf{D}\boldsymbol{u} \text{ in } L_M \quad \text{and} \quad \mathbf{S}(x,\mathbf{D}\boldsymbol{u}^n) \overset{*}{\rightharpoonup} \chi \text{ in } L_{M^*}$$
- Since $\mathbf{D}\boldsymbol{u}^n \in L_M$ and $L_M \neq E_M$, $\mathbf{w} = \mathbf{D}\boldsymbol{u} + h\boldsymbol{v} \in L_M$ is not a proper test function
- Monotonicity methods and Browder-Minty trick adapted to non-reflexive Orlicz spaces:

  $$\mathbf{w} = \mathbf{D}\boldsymbol{u}\mathbb{1}_{\{|\mathbf{D}\boldsymbol{u}|<j\}} + h\boldsymbol{v}\mathbb{1}_{\{|\mathbf{D}\boldsymbol{u}|<i\}}, \;\; \boldsymbol{v} \in L^\infty$$

  (Mustonen, Tienari (1999))

Integration by parts

- How to obtain

$$\limsup_{n\to\infty} \int \mathbf{S}(x, \mathbf{D}\boldsymbol{u}^n) : \mathbf{D}\boldsymbol{u}^n \leqslant \int \boldsymbol{\chi} : \mathbf{D}\boldsymbol{u} \quad ?$$

Integration by parts

- How to obtain

$$\limsup_{n\to\infty} \int \mathbf{S}(x, \mathbf{D}\boldsymbol{u}^n) : \mathbf{D}\boldsymbol{u}^n \leqslant \int \boldsymbol{\chi} : \mathbf{D}\boldsymbol{u} \quad ?$$

- Is a solution an admissible test function?

$$\partial_t \boldsymbol{u}^n - \operatorname{div}_x \mathbf{S}(x, \mathbf{D}\boldsymbol{u}^n) + \cdots = \boldsymbol{f}$$

Integration by parts

- How to obtain

$$\limsup_{n\to\infty} \int \mathbf{S}(x, \mathbf{D}\boldsymbol{u}^n) : \mathbf{D}\boldsymbol{u}^n \leqslant \int \boldsymbol{\chi} : \mathbf{D}\boldsymbol{u} \quad ?$$

- Is a solution an admissible test function?

$$\partial_t \boldsymbol{u}^n - \operatorname{div}_x \mathbf{S}(x, \mathbf{D}\boldsymbol{u}^n) + \cdots = \boldsymbol{f}$$

- How to obtain the integration by parts formula?

Integration by parts

- How to obtain

$$\limsup_{n\to\infty} \int \mathbf{S}(x, \mathbf{D}\boldsymbol{u}^n) : \mathbf{D}\boldsymbol{u}^n \leqslant \int \boldsymbol{\chi} : \mathbf{D}\boldsymbol{u} \quad ?$$

- Is a solution an admissible test function?

$$\partial_t \boldsymbol{u}^n - \mathrm{div}_x\, \mathbf{S}(x, \mathbf{D}\boldsymbol{u}^n) + \cdots = \boldsymbol{f}$$

- How to obtain the integration by parts formula?

  Classical case: $t_0, t_1 \in (0, T)$, $u \in L^p(0, T; X)$, $u_t \in L^{p'}(0, T; X^*)$, X reflexive and separable Banach space and $X \subset H = H^* \subset X^*$, then

$$\int_{t_0}^{t_1} \langle u_t, u \rangle_{X, X^*}\, dt = \frac{1}{2}\|u(t_1)\|_H^2 - \frac{1}{2}\|u(t_0)\|_H^2$$

Integration by parts

- How to obtain

$$\limsup_{n\to\infty} \int \mathbf{S}(x, \mathbf{D}\boldsymbol{u}^n) : \mathbf{D}\boldsymbol{u}^n \leqslant \int \boldsymbol{\chi} : \mathbf{D}\boldsymbol{u} \quad ?$$

- Is a solution an admissible test function?

$$\partial_t \boldsymbol{u}^n - \mathrm{div}_x\, \mathbf{S}(x, \mathbf{D}\boldsymbol{u}^n) + \cdots = \boldsymbol{f}$$

- How to obtain the integration by parts formula?

  Classical case: $t_0, t_1 \in (0, T)$, $u \in L^p(0, T; X)$, $u_t \in L^{p'}(0, T; X^*)$, X reflexive and separable Banach space and $X \subset H = H^* \subset X^*$, then

$$\int_{t_0}^{t_1} \langle u_t, u\rangle_{X,X^*}\, dt = \frac{1}{2}\|u(t_1)\|_H^2 - \frac{1}{2}\|u(t_0)\|_H^2$$

- Lack of density of smooth functions in L_M and $L_M(0, T; L_M(\Omega)) \neq L_M((0, T) \times \Omega)$

Some problems – admissible test function, integration by parts formula

- How to obtain integration by parts formula?

Some problems – admissible test function, integration by parts formula

- How to obtain integration by parts formula?
-
$$\int_{s_0}^{s}\int_{\Omega} \boldsymbol{u}\cdot\partial_t(\varphi^j) = \int_0^T\int_{\Omega} \boldsymbol{\chi}\cdot\mathbf{D}\varphi^j - \int_0^T\int_{\Omega} \boldsymbol{f}\cdot\varphi^j + \dots .$$
- If M^* satisfies Δ_2–condition, then $\boldsymbol{\chi}\in L_{M^*} = E_{M^*}$ and we pass to the limit using weak* convergence,
  i.e. $\mathbf{D}\varphi^j \overset{*}{\rightharpoonup} \mathbf{D}\boldsymbol{u}$ in L_M

Some problems – admissible test function, integration by parts formula

- How to obtain integration by parts formula?
- $$\int_{s_0}^{s}\int_\Omega \boldsymbol{u}\cdot\partial_t(\varphi^j) = \int_0^T\int_\Omega \boldsymbol{\chi}\cdot \mathbf{D}\varphi^j - \int_0^T\int_\Omega \boldsymbol{f}\cdot\varphi^j + \dots.$$
- If M^* satisfies Δ_2–condition, then $\boldsymbol{\chi}\in L_{M^*} = E_{M^*}$ and we pass to the limit using weak* convergence,
  i.e. $\mathbf{D}\varphi^j \xrightarrow{*} \mathbf{D}\boldsymbol{u}$ in L_M
- If M^* does not satisfy Δ_2, we can use a modular convergence
- z^j converges to z in modular in L_M if there exists $\lambda > 0$, such that

$$\int_Q M(x,(z^j - z)/\lambda)dxdt \to 0$$

Some problems – admissible test function, integration by parts formula

- How to obtain integration by parts formula?
- $$\int_{s_0}^{s}\int_{\Omega} \boldsymbol{u}\cdot\partial_t(\varphi^j) = \int_0^T\int_{\Omega} \boldsymbol{\chi}\cdot \mathbf{D}\varphi^j - \int_0^T\int_{\Omega} \boldsymbol{f}\cdot\varphi^j + \dots.$$
- If M^* satisfies Δ_2–condition, then $\boldsymbol{\chi} \in L_{M^*} = E_{M^*}$ and we pass to the limit using weak* convergence,
  i.e. $\mathbf{D}\varphi^j \overset{*}{\rightharpoonup} \mathbf{D}\boldsymbol{u}$ in L_M
- If M^* does not satisfy Δ_2, we can use a modular convergence
- z^j converges to z in modular in L_M if there exists $\lambda > 0$, such that
  $$\int_Q M(x,(z^j - z)/\lambda)dxdt \to 0$$
- The closure of smooth functions w.r.t. weak* and modular topology of symmetric gradients needs to coincide.

Some problems – Korn inequality

Some problems – Korn inequality

- Is the Korn inequality satisfied?

$$\|\nabla \boldsymbol{u}\|_{L_M} \leqslant c\|\mathbf{D}\boldsymbol{u}\|_{L_M}$$

- Fuchs: Δ_2 for M and M^* is needed

Some problems – Korn inequality

- Is the Korn inequality satisfied?

$$\|\nabla \boldsymbol{u}\|_{L_M} \leqslant c\|\mathbf{D}\boldsymbol{u}\|_{L_M}$$

- Fuchs: Δ_2 for M and M^* is needed
- Strauss:

$$\|\boldsymbol{u}\|_{L^{\frac{d}{d-1}}(\Omega)} \leqslant c_d\|\mathbf{D}\boldsymbol{u}\|_{L^1(\Omega)}$$

Some problems – Korn inequality

- Is the Korn inequality satisfied?

$$\|\nabla \boldsymbol{u}\|_{L_M} \leqslant c\|\mathbf{D}\boldsymbol{u}\|_{L_M}$$

- Fuchs: Δ_2 for M and M^* is needed
- Strauss:

$$\|\boldsymbol{u}\|_{L^{\frac{d}{d-1}}(\Omega)} \leqslant c_d\|\mathbf{D}\boldsymbol{u}\|_{L^1(\Omega)}$$

- If M is isotropic and homogenous

$$\|M(|\boldsymbol{u}|)\|_{L^{\frac{d}{d-1}}(\Omega)} \leqslant c_d\|M(c_\Omega|\mathbf{D}\boldsymbol{u}|)\|_{L^1(\Omega)}$$

Topologies in Musielak-Orlicz spaces

Topologies in Musielak-Orlicz spaces

- strong – norm topology

$$\|z^j - z\|_{L_M} \longrightarrow 0$$

Topologies in Musielak-Orlicz spaces

- strong – norm topology
$$\|z^j - z\|_{L_M} \to 0$$

- modular topology
z^j converges to z in modular in L_M if there exists $\lambda > 0$, such that
$$\int_Q M(x, (z^j - z)/\lambda) dxdt \to 0$$

Topologies in Musielak-Orlicz spaces

- strong – norm topology

$$\|z^j - z\|_{L_M} \to 0$$

- modular topology
  z^j converges to z in modular in L_M if there exists $\lambda > 0$, such that

$$\int_Q M(x, (z^j - z)/\lambda) dxdt \to 0$$

- weak-*

$$\sigma(L_M, E_{M^*})$$

Bibliography

- J. Musielak, M.S. Skaff, M.M. Rao, Z.D. Ren,
- J. Nečas, J. Málek, M. Bulíček, J. Frehse, L. Diening, J. Wolf, M. Ružička, M. Fuchs, P. Kaplicky
- J.-P. Gossez, T. Donaldson, V. Mustonen,
- A. Cianchi,
- P. Gwiazda, A. Świerczewska-Gwiazda,
- F. Klawe, M. Kalousek
- E. Emmrich
- R. DiPerna, P.L. Lions, F. Murat, L. Boccardo,
- P. Wittbold, A. Zimmermann
- I. Chlebicka, A. Zatorska-Goldstein, C. De Filippis
- J. Skrzeczkowski, J. Woźnicki,
- ...

Bibliography

- J. Musielak, M.S. Skaff, M.M. Rao, Z.D. Ren,
- J. Nečas, J. Málek, M. Bulíček, J. Frehse, L. Diening, J. Wolf, M. Ružička, M. Fuchs, P. Kaplicky
- J.-P. Gossez, T. Donaldson, V. Mustonen,
- A. Cianchi,
- P. Gwiazda, A. Świerczewska-Gwiazda,
- F. Klawe, M. Kalousek
- E. Emmrich
- R. DiPerna, P.L. Lions, F. Murat, L. Boccardo,
- P. Wittbold, A. Zimmermann
- I. Chlebicka, A. Zatorska-Goldstein, C. De Filippis
- J. Skrzeczkowski, J. Woźnicki,
- ...
- Partial Differential Equations in Anisotropic Musielak-Orlicz Spaces, I.Chlebicka, P. Gwiazda, A. Świerczewska-Gwiazda, A. W-K, Springer, 2021

Thank you!